

Lampreys play a complicated role in their ecosystems. Their ability to adjust their sex ratio in response to food availability and possibly other factors adds much complexity to understanding their behavior and how this can affect their ecosystem, which presents a unique challenge for researchers. This underscores the need for a comprehensive understanding of these adaptive mechanisms, as they play a pivotal role in shaping the broader ecological landscape.

In this year's MCM competition, our team developed a probabilistic model to simulate the population dynamics of lampreys in a specific habitat. We assert that the population of male and female lampreys can be effectively modeled using a system of differential equations whose variables depend on several parameters, including birth rate, mortality rate, and the probability of a newborn being female. These parameters, in turn, are influenced by external factors such as water temperature, predator abundance, and food availability within the habitat.

Utilizing our model, we conduct extensive simulations to analyze the long-term population dynamics of lampreys. Our investigation delves into the intricate interactions between male and female lampreys, examining their effects on predator numbers and food availability within the habitat. To gauge the model's robustness, we subject it to sensitivity analysis by adjusting environmental parameters and introducing random conditions to simulate real-world variability.

Lastly, employing our model, we contrast the population dynamics of species with a flexible sex-determination mechanism against those with a fixed sex ratio. We simulate the population dynamics of different species through 100 different simulations with changing habitat conditions. Our findings indicate that although species capable of adjusting their sex ratio experience advantages such as increased food availability and reduced predator numbers, they tend to exhibit lower birth rates and higher mortality rates. Consequently, this results in an overall lower population level compared to species with a fixed sex ratio. On a higher level, we conclude that lampreys' capacity to alter sex not only enables swift adaptation to local conditions, leading to increased food availability, but also creates ecological space, enhancing the potential for symbiotic relationships and stability among coexisting species.

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1 Introduction

Sea lampreys, a species native to the Atlantic Ocean, are a parasitic fish which rely on the blood and body fluids of other fish to survive, which possess an unusual feature as a population– volatile sex ratios deviating far from even. Studies show that such fluctuations largely depend on environmental factors affecting the lampreys’ food availability [1]. We are interested in investigating this intriguing phenomenon and its role in the ecosystems in which sea lampreys exist. In particular, we would like to develop a model which can describe and predict the altering of sea lamprey populations’ sex ratios, and, furthermore, how these variations will influence their environment. Studying sea lampreys will provide us with an insight into the more general inquiry concerning species which have the ability to alter sex ratio, and the various ways this ability can impact the dynamics of ecosystems.

1.1 Background Information

There have been numerous studies delving into the mechanisms of sex determination in sea lampreys, and many observe a correlation between unproductive environments having more male-skewed populations and vice versa, suggesting a connection between growth rate and the sex determination process. Many credible institutions have published research reports and experiments dedicated to the assessment of this matter, including the National Library of Medicine [1] and the Great Lakes Fishery Commission [2]. The study of the population dynamics of sea lampreys have been conducted quite a lot in the past in order to develop more effective population-control strategies. These studies provide imperative understanding into how the population of sea lampreys fluctuate in response to changing environmental conditions [3]. However, existing bodies of research study the point in question within experimental frameworks, analyzing observed changes in sex ratios based on controlling and changing environmental factors, lacking transferable results that can be used to model changes in population dynamics. In our analysis, we aim to provide mathematically-driven solutions which align with existing observations while also generating visions into adjustable conditions and scenarios.

2 Assumptions

When faced with limited data and time constraints, making certain assumptions becomes a practical necessity to render a model feasible and manageable. These assumptions serve as simplifications or constraints that enable the development and execution of the model within the available resources. To guarantee the feasibility of our model, we make the following assumptions:

2.1 Mortality Rate is the Same between Female and Male Lampreys

This assumption is reasonable because, in most species, unless there is a specific reason (such as sexual dimorphism or differences in behavior), mortality rates tend to be relatively consistent between genders. It simplifies the analysis by treating male and female lampreys as having equal susceptibility to mortality factors, streamlining the modeling process.

2.2 Context Factors are Predictable

This assumption is essential for experimental design and controlled studies where researchers can isolate the impact of specific factors on the ecosystem, making it a controlled and predictable environment.

On the other hand, in reality, the causes of contextual changes within an ecosystem are likely to be complex and multifaceted. Ecosystems are dynamic and subject to a multitude of interacting variables that can have both direct and indirect effects on their conditions.

2.3 Prey and Predators are Independent Species within the Ecosystem

In other words, changes in population dynamics within predators will not directly affect that within their prey.

However, in most cases, prey and predators are interdependent species within ecosystems, forming vital trophic relationships that regulate populations, maintain biodiversity, and contribute to ecosystem services [4]. Their dynamic interactions involve cascading effects, coevolution, and adaptation, influencing the overall health and resilience of ecosystems.

2.4 Carrying Capacity is Predictable

We are able to calculate a number which corresponds to the carrying capacity of a population, using a resource-based model in which we solve for the maximum population size any given ecosystem can support. It assumes that the factors influencing population growth and resource availability can be accurately represented in the model.

Alternatively, there exist other models that might provide a more precise representation of carrying capacity, and one notable example is the Lotka–Volterra equations [5]. These equations, rooted in ecological dynamics, consider the interplay between predator and prey populations and offer a more nuanced perspective on the factors influencing carrying capacity.

2.5 Characterization of time

We can model the interactions and changes occurring within the ecosystem in terms of time periods, where each change to a parameter maintains its status within a given time period. While this deviates from the continuous nature of time as an independent variable on which the rate of change of our parameters depends, our time periods are short enough to simulate the conditions in a continuous model in which interactions and inflicted changes happen simultaneously.

3 Model

3.1 Introduction of Variables

We shall formulate the problem as a system of differential equations. Hence, we first define some variables and parameters as follows:

- $N_F \in [0, \infty]$: Total population of females.
- $N_M \in [0, \infty]$: Total population of males.
- $D_F \in [0, 1]$: Ratio of females over the total population, computed as below

$$D_F = \frac{N_F}{N_F + N_M}.$$

- $F \in [-\infty, \infty]$: Food availability, assessing the sufficiency of resources for a given population. A negative number with large magnitude would mean that the current amount of food resources available is quite limited relative to the size of the population.
- $C \in [-\infty, \infty]$: Contextual factors, an overall score considering the following four environmental factors:
 - $C_T \in [-1, 1]$: Temperature (-1 and 1 are extreme temperatures, and increasing values within this range correspond linearly to increasing temperature).

- $C_W \in [-1, 1]$: Water acidity (-1 and 1 correspond to the extremities of acidity and alkalinity, and increasing values within this range correspond linearly to increasing acidity).
- $C_H \in [-1, 1]$: Human factors, computed as a linear combination of temperature and water acidity (-1 corresponds to rare human activities which may cause population to grow beyond carrying capacities, 1 corresponds to too many human activities, potentially causing populations to decline, and increasing values within this range correspond linearly to increasing amount of human activities).

$$C_H = 2(0.8|C_T| + 0.2|C_W|) - 1$$

Note we assign the weights to be 0.8 and 0.2 for C_T and C_W respectively, as we assume that temperature causes a greater influence on human behaviors compared to water acidity, roughly with the ration 8 : 2.

- $C_O \in [-1, 1]$: Other factors contributing to changes in population (-1 corresponds to an extreme and 1 corresponds to the other).

Note we take the values of the above four variables to be in the range $[-1, 1]$ in order to standardize our ratings of these parameters into a uniform scale which is intuitive in meaning (i.e. extreme values in both directions correspond to extremities in the measured properties of the environment).

We later convert these ratings into $C \in [-\infty, \infty]$ in section 3.3.1, which is later used to compute the probability that a newborn is female, P_F , in section 3.3.6.

- $P_F \in [0, 1]$: Probability that a newborn is female.
- $M \in [0, 1]$: Mortality rate, representing the percentage of the overall population expected to succumb within a given time frame.
- $B \in [0, 1]$: Birth rate concerning the entire population. Similar to how we treat mortality, we do not distinguish the birth rate between female and male lampreys in R_B .
- $c_P \in [0, 1]$: Predator-prey ratio threshold for lamprey, above which the ratio of predators' population to lampreys' population is considered as too high concerning the survival of lampreys, and vice versa.
- $H \in [-1, 1]$: Relative size of predator population, compared to lampreys' population. We streamline the logic under which we set the values of contextual factors and apply that to the variable H . A value of -1 corresponds to the predator population being larger than a value which allows sustained survival of the lampreys, and a value of 1 corresponding to the opposite. Finally, 0 means that the predator population is around an ideal ratio with the lamprey population, contributing to their sustained survival.

3.2 Model Formulation

Our entire model of the population of lampreys follows the following system of differential equations:

$$\begin{aligned} \frac{dN_F}{dt} &= R_B \cdot P_F - R_M \\ \frac{dN_m}{dt} &= R_B \cdot (1 - P_F) - R_M \end{aligned}$$

While this is quite a straightforward model, it mostly captures the population change of most species, since most population sizes only change based off of the birth and mortality rate, and we apply an extra probability representing that lampreys may change their probability of being different sexes. Each of these rates and probabilities are then calculated through other sub-models.

3.3 Computation of Variables

3.3.1 Context factor C

First we present the formula of the area corresponding to the value of C that we are interested, denoted as C_{area} , as the linear combination of the four factors:

$$C_{\text{area}} = 1 - (\alpha_1|C_T| + \alpha_2|C_W| + \alpha_3|C_H| + \alpha_4|C_O|),$$

for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$.

Note C_{area} is a value between 0 and 1. We construct C_{area} this way since for each of C_T , C_W , C_H , and C_O , the more they deviate from 0, the worse their contextual representation are for the survival of lampreys. Hence, C_{area} , formulated as the difference between 1 and a linear combination of the four factors, tends to 1 when contextual conditions are ideally-suited for lampreys' survival, and 0 otherwise.

Then, we compute C as the x -value corresponding to C_{area} under the standard normal distribution, which has mean 0 and standard deviation 1. Consistent with the trend we have in C_{area} , C tends to infinity under ideal conditions, and vice versa.

3.3.2 Food availability F

We compute the food sufficiency as follow

$$F = \frac{C}{\gamma_1 N_M + \gamma_2 N_F},$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ represent male lampreys' and female lampreys' consumption of resources, respectively. We structured the variable F in this manner due to its correlation with contextual factors, indicating a positive association with overall resource abundance. However, it is important to note that in the presence of a large population, heightened competition for resources may lead to a decrease in food availability. Therefore, in the denominator, we incorporate a linear combination of the male and female populations. Additionally, we introduce γ_1 and γ_2 to differentiate the food consumption abilities between the two genders.

3.3.3 Birth rate B

Birth rate is formulated as

$$B = \lambda D_F C_{\text{area}},$$

for some constant $\lambda \in \mathbb{R}_{>0}$. The formulation of birth rate is that it is positively correlated with the product of the female ratio and the corresponding area of the context factors.

3.3.4 Relative size of predator population H

We first compute the total number of predators of lampreys N_P , as follows

$$N_P = \frac{L}{1 + b e^{-k(N_M + N_F)}},$$

where L is the carrying capacity [6] of the predator population, and $b, k \in \mathbb{R}$ are constants. This equation is derived from Verhulst's logistic equation involving the carrying capacity of a species within an ecosystem [7].

Then, we are ready to compute the relative size of predator population as

$$H = \frac{1}{\frac{x}{2c_P} + 0.5} - 1,$$

where $x = \frac{N_P}{N_M + N_F}$ represents the ratio of predators over the total population of lampreys. Note that when $x = c_P$, $H = 0$, consistent with our definition of c_P in section 3.1. Also, H goes to 1 with lower ratios and -1 with higher ratios.

3.3.5 Mortality rate M

The mortality rate is formulated as

$$M = \frac{\mu}{H' C_{\text{area}}},$$

for $H' = \frac{H+1}{2}$ and some constant $\mu \in \mathbb{R}_{>0}$. The intuition is that we introduce a new variable H' which manipulates the value of H to make its range $[0, 1]$ so that the scale is consistent with that of mortality rate. In particular, values of H' near 0 represent too many predators, and those near 1 represent extremely few predators.

Hence, the mortality rate should be negatively correlated to the product of H' and C_{area} , since larger values of the product represents better environments for the lampreys concerning survival.

3.3.6 Probability that a newborn is female P_F

We model P_F using a logistic regression with inputs food sufficiency F , contextual factors C , and female proportion D_F . Note we decided to use logistic regression, because it is a widely employed mathematical method, especially for probability prediction. The logistic function transforms the linear combination of predictors into probabilities, ensuring that the predictions smoothly transition from 0 to 1. The sigmoidal shape is crucial for avoiding unrealistic extreme values in probability estimates. Here, P_F is formulated as a logistic function with arguments food F , context factors C , and female ratio D_F :

$$P_F = \frac{1}{1 + e^{-(\beta_0 + \beta_1 F + \beta_2 C + \beta_3 D_F)}}$$

for constants $\beta_0, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$.

Note we need to make sure $P_F = D$ when $C = F = 0$, since if environmental factors remain neutral (i.e. water not too acidic or too alkaline, temperature not too high or too low, etc.), and food resources are sufficient enough to support the population, then the probability that a newborn is female should be roughly the same as the current proportion of females in the population.

Substituting the values, we obtain $-(\beta_0 + \beta_3 D_F) = \ln\left(\frac{1-D}{D}\right)$. Therefore, when we plug in these values, we get:

$$P_F = \frac{1}{1 + e^{-(\beta_1 F + \beta_2 C + \ln\left(\frac{1-D}{D}\right))}} = \frac{1}{1 + \left(\frac{1-D}{D}\right) e^{-\beta_2 F} e^{-\beta_3 C}}.$$

4 Simulation of Model

Notice that due to the complexity of the interactions within our differential equations, it is quite challenging to find a closed form solution for our model. Hence, we shall simulate it over a long period of time and determine its long-term behavior.

4.1 Algorithm Pseudo-code

Since we formulate the model as a probability chain, we will specify the order in which we update variables, and provide our reasoning as to how such decisions were made and what type of updates occur in each step.

1. We first update factors concerning the environmental context, which include temperature C_T , water acidity C_W , human factors C_H , and other factors C_O . Note that our determination of these factor values reflect our assessment of reasonable changes caused by seasonality and human activities.
2. Then we update the relative size of the predator population, H , using the updated values for contextual factors to compute changes in the relative size of predator populations.
3. We update the total populations of female and male lampreys, respectively, since the population of sea lampreys are affected by changes to their habitats and predator-prey interactions.
4. Using the updated population of female and male lampreys and the updated context factors, we now update food availability, F , which is, according to prior research, the prominent factor influencing sex determination in sea lampreys.
5. Then, we update P_F , the new probability that a newborn sea lamprey is female, using contextual factors and the population's current proportion which is of the female sex.
6. We update B , the overall birthrate of sea lampreys in a given time period, which we model as being positively correlated to the current proportion of female lampreys and contextual factors.
7. Lastly, update M , the mortality rate of sea lampreys in a given time period. After determining the above values, we can then calculate the total population of sea lampreys at the end of the corresponding time period, using B and P_F to calculate the number of females born and B and M to calculate the overall change in population.

For our calculations, we shall include a momentum constant m such that the values are smoother in other words, when we calculate a value, we take m times the new value and $1 - m$ times the old value, and combine them in order to substitute them into our original values. The only variables that do not contain this momentum are the male and female populations, which are determined by the birth and mortality rate, as well as the probability of being female.

4.2 Baseline Model

We first fine-tune a baseline model, denoted as *experiment 0*, where we let $C_T, C_W, C_H, C_O = 0.5$, $N_m, N_f = 50,000$, and $m = 0.2$. In other words, our baseline model describes an ecosystem with moderate conditions, which intuitively should result in population dynamics which converge to a stable sex ratio and relatively mild fluctuations.

We demonstrate the behavior of the model below:

4.2.1 Male, Female, and the Total Population

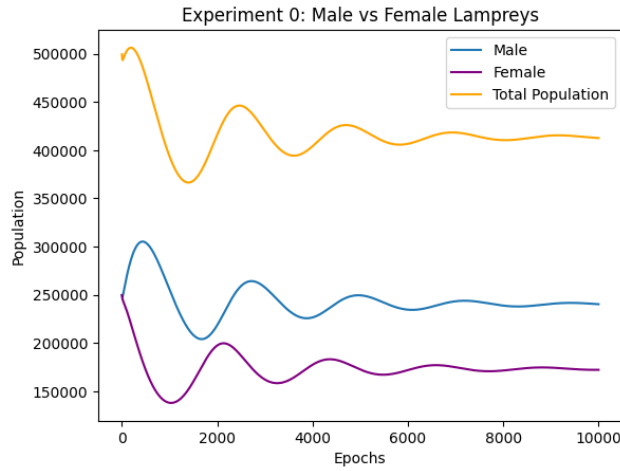


Figure 1: Population of lampreys

In figure 1 we can see the behavior of the sea lamprey population as a whole, as well as broken down by sex. Both sexes oscillate initially in a manner which roughly complements the other sex (i.e. male populations increase as female populations change in the other direction). Eventually, however, both sexes converge at a stable equilibrium. This, in turn, means that the total population also stabilizes eventually. We can see that this aligns with theoretical bases for long-term population behavior with unchanged environmental factors, as our parameters are set to amounts sufficient enough to support the lampreys' survival, while stationary environmental conditions should theoretically stabilize the populations around a steady value.

4.2.2 Number of Predators

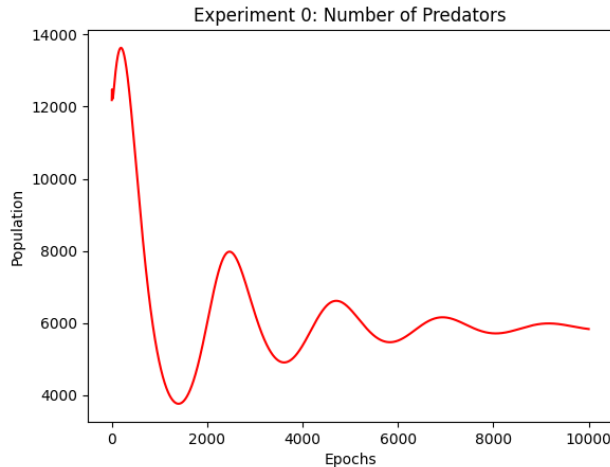


Figure 2: Number of predators

Figure 2 illustrates the change in predator population over time, given unchanged moderate environmental conditions. Again, similar to the theoretical reasoning for figure 1, we observe oscillations

in the population, as we consider sea lampreys to be the primary food source of predators, along with environmental factors simultaneously affecting all facets of the ecosystem. Thus the eventual stabilization of the predator population aligns with trends we observe in figure 1.

4.2.3 Probability that a Newborn is Female

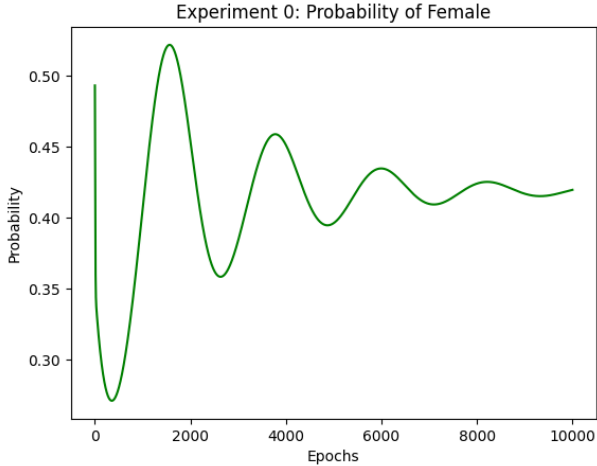


Figure 3: Probability that a new born is female

Figure 3 illustrates the trends in the change of the probability that a newborn sea lamprey is female over time, beginning with $P_F = 0.5$, and fluctuating within ranges of values lower than 0.5. This observation is supported by our existing knowledge surrounding the gender composition of sea lamprey populations [1], as female lampreys have been observed to consistently comprise a lower proportion of total population. Thus it is reasonable that we see an eventual stabilization around the value of 0.3, meaning that under medium conditions on our defined scales, female lampreys will, in time, approximately make up 30 percent of the entire sea lamprey population.

4.2.4 Food Availability

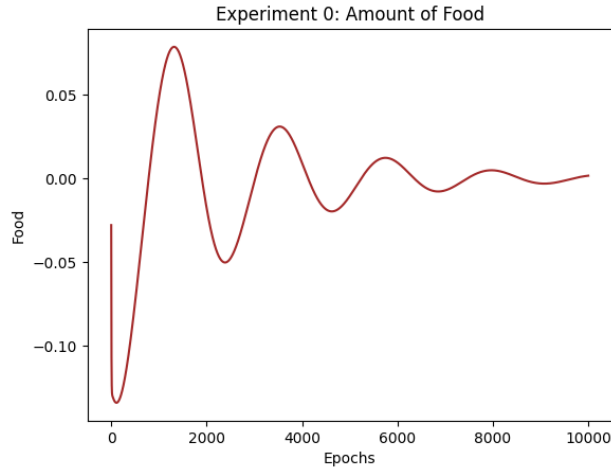


Figure 4: Food availability

The trend in the change of food availability in figure 4 reasonably corresponds to the changes observed previously in lamprey populations, assuming stationary contextual parameters. Recall that our formula for determining food availability modeled its dependent relationship on environmental conditions and inversely to lamprey population, thus it is justified that we see oscillations which are inverse in direction to those of the population variables.

4.2.5 Birth Rate

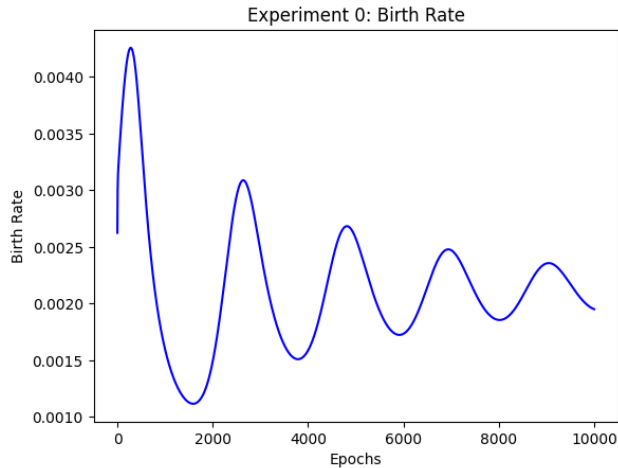


Figure 5: Number of Predators

We model the daily birth rate of sea lamprey to be positively correlated with the ratio of female lampreys to total population, as well as the aggregate of contextual factors which we consider to affect spawning activities of lamprey. Note in figure 5 it oscillates heavily around small x values and tend to be stable around the value of 0.0020 starting from around the 8000-th iteration. Furthermore, the

trend of birth rate is coherent with the changes observed in the probability of a newborn sea lamprey being female.

4.2.6 Mortality Rate

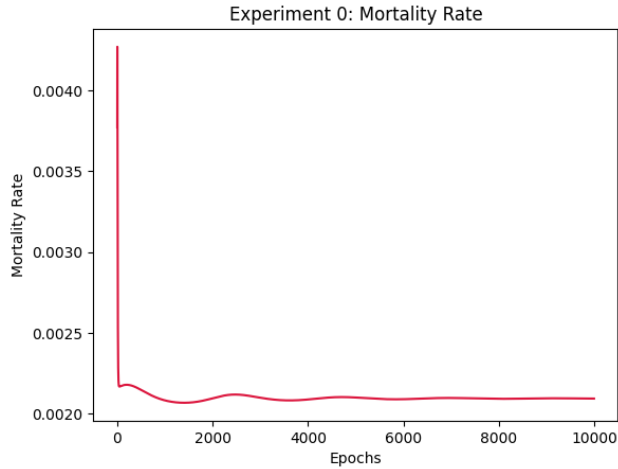


Figure 6: Mortality Rate

Lastly, we analyze the changes in mortality rate observed in our baseline model. Since we model the mortality of sea lamprey to be primarily affected by the presence of predators and local conditions, when contextual factors remain unchanged, the survival or otherwise of sea lamprey should theoretically be entirely dependent on the population of predators. Here we observe a spike in mortality initially, corresponding to the initial increase in predator population, and a relatively swift convergence to a stable value of around 0.0020, aligning with the stabilization in the number of predators.

4.3 Sensitivity Analysis

Owing to the numerous constants involved, even minor alterations in these values could potentially lead to significant shifts in population behaviors. Consequently, our approach involves isolating individual variables and examining their impact on outcomes. Given time and length constraints, our primary focus is on studying the female, male, and overall populations of lampreys, along with the probability of a newborn being female.

4.3.1 Context C

We shall first consider the cases where the contextual factors are more and less than optimal, whereas the other factors remain the same as in the baseline model:

1. $C_T, C_W, C_H, C_O = 0.8$, $N_m, N_f = 50,000$, and $m = 0.2$. In other words, we study the case where contexts are worse than the neutral scenario.

Note that a value of 0.8 for contextual factors signals that environmental conditions deviate far from ideal conditions which support the sustained survival of sea lampreys. Under these conditions, we observe a rather agile descent into extinction.

Now observe the change in the proportion of newborns which are expected to be female in figure 8.

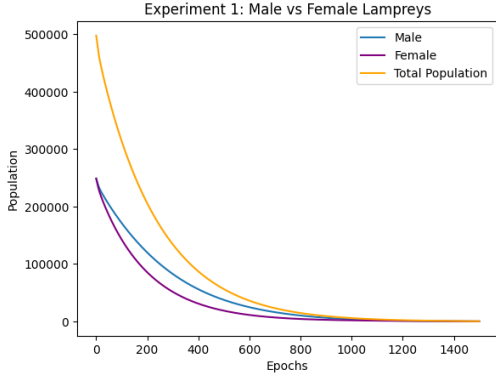


Figure 7: Population (exp. 1)

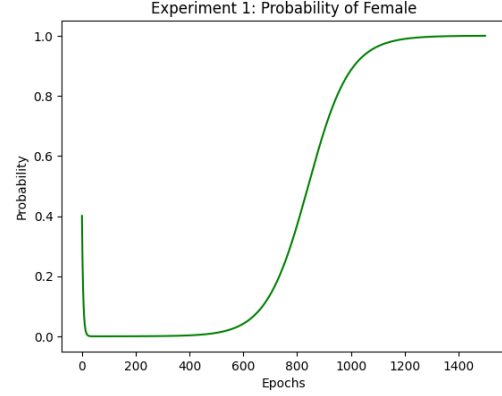


Figure 8: Probability (exp. 1)

Based on this change in parameters, the probability of a newborn being female resembles the shape of a logistical growth model, plateauing near the value of 1. Since the total population of lampreys quickly decreased under these conditions, the relative food availability to support sex determination would be higher (as we observe a rapid decay of total population), thus contributing to an increase in the probability that larval-stage lamprey becomes female.

2. $C_T, C_W, C_H, C_O = 0.2$, $N_m, N_f = 50,000$, and $m = 0.2$. In other words, we study the case where contexts are better than the neutral scenario.

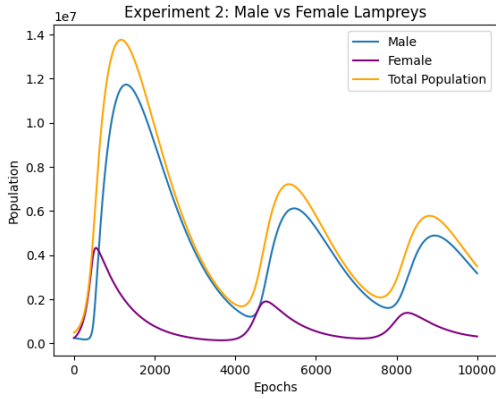


Figure 9: Population (exp. 2)

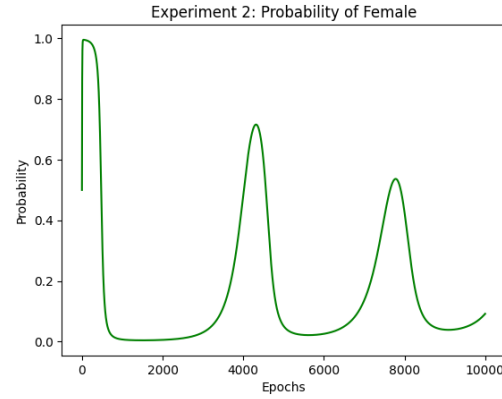


Figure 10: Probability (exp. 2)

A context value of 0.2 presents the results illustrated in figures 9 and 10. We observe an oscillation with higher variance than that in the baseline model, and much wider fluctuations in the probability of a sea lamprey being female. However, there is evidently an overall trend of stabilization, as we see in figure 9, the populations of male and female lampreys are getting closer and closer to each other with time.

3. C_T and C_W are random walks between -1 and 1 starting at 0.5 , C_O is constant, $C_H = 2(0.8|C_T| + 0.2|C_W|) - 1$, $N_m, N_f = 50,000$, and $m = 0.2$. In other words, it is the case where temperature and water acidity are random.

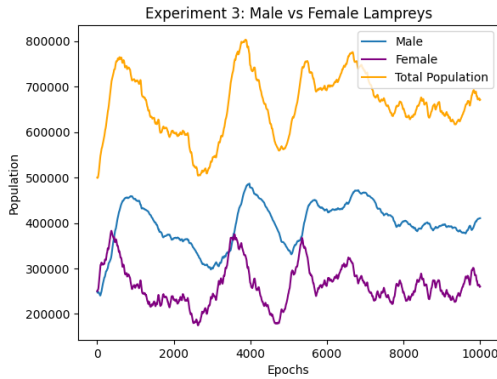


Figure 11: Population (exp. 3)

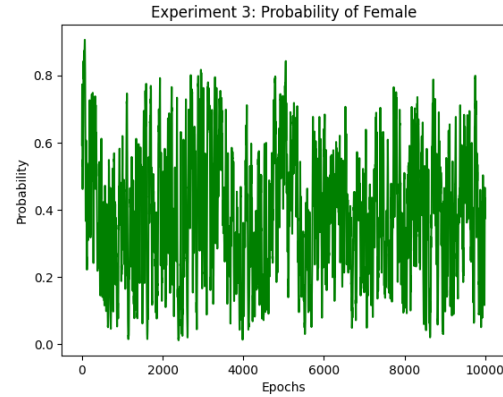


Figure 12: Probability (exp. 3)

Please note that figure 19 depicts patterns in the male, female, and overall lamprey populations, resembling those observed in prior experiments. However, these trends exhibit less predictability, with curves that are not as smooth. In contrast, figure 20 displays distinctly different patterns. Here, the probability of a newborn being female fluctuates significantly, oscillating widely between values of 0 and 1.

4.3.2 Momentum

In section 4.1, we defined momentum, denoted by m , as the proportion in which the new calculated value consists in our next iteration of the variable. We then test the sensitivity of the parameter below, while keeping the rest of the parameters the same as the baseline model.

1. $m = 0.01$, other factors the same as in the baseline model.

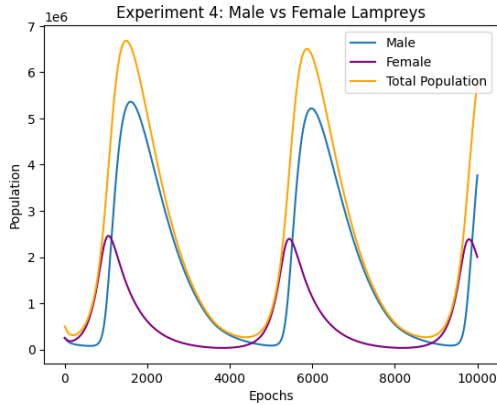


Figure 13: Population (exp. 4)

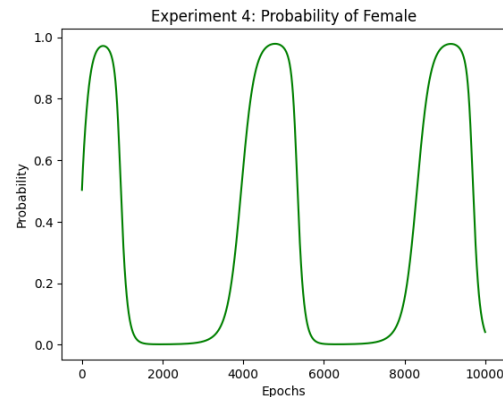


Figure 14: Probability (exp. 4)

2. $m = 0.99$, other factors the same as in the baseline model.

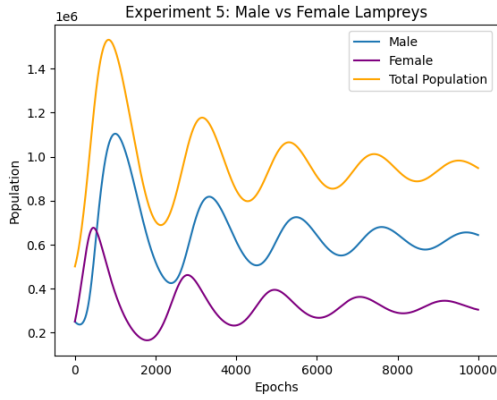


Figure 15: Population (exp. 5)

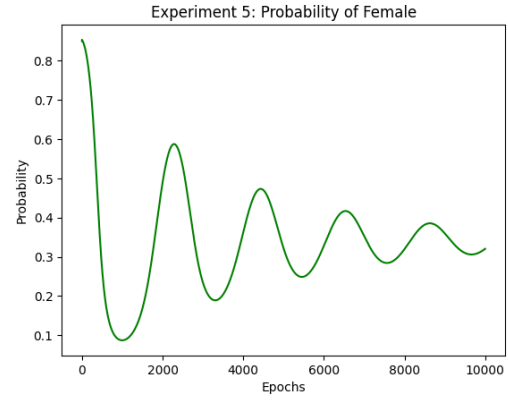


Figure 16: Probability (exp. 5)

Our first observation is that both values follow similar patterns as the baseline model. This is expected since we are only changing the rate in which the variables are updated rather than the values themselves. We then notice that the graphs with a smaller m tends to have longer rises and falls, which makes sense, since we are keeping most of the initial momentum in our values, and the newly calculated rates have less of an effect on changing the direction.

4.3.3 Predator-prey ratio threshold

We then test the predator-prey ratio threshold c_P . Recall that we defined this value to be the predator-prey population ratio, above which we considered the number of predators too high compared to the lamprey population for their sustained survival.

1. $c_P = 0.1$, other factors the same as in the baseline model.

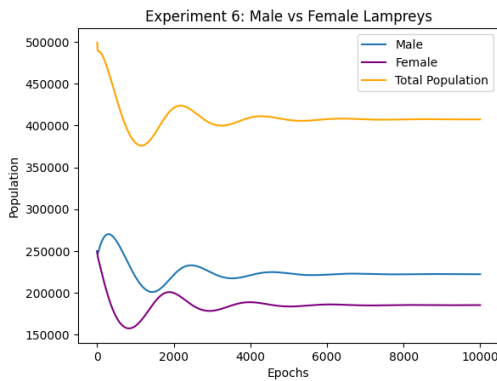


Figure 17: Population (exp. 6)

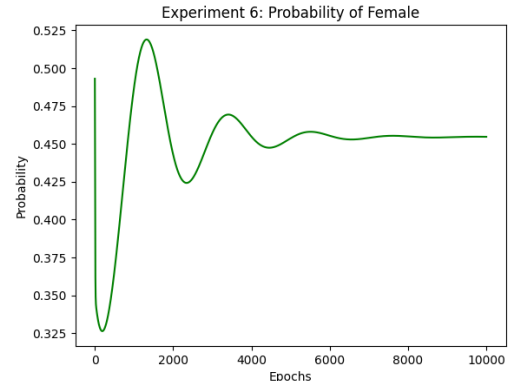


Figure 18: Probability (exp. 6)

2. $c_P=0.9$:

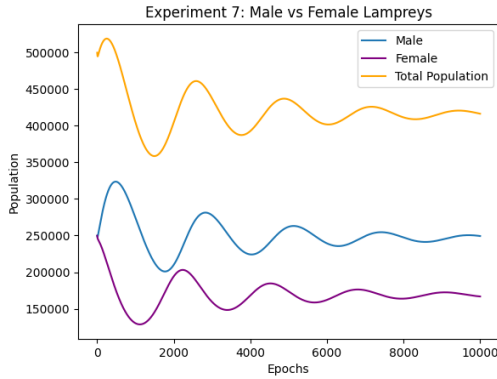


Figure 19: Population (exp. 7)

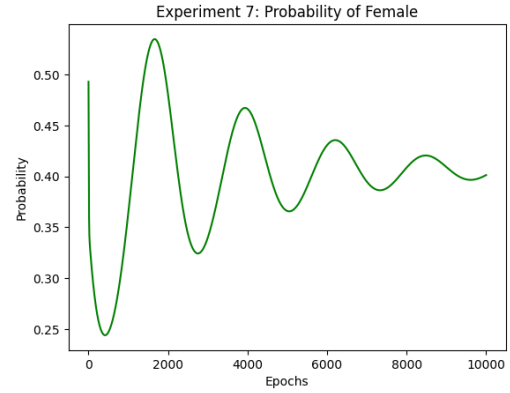


Figure 20: Probability (exp. 7)

5 Lampreys' Ability to Alter the Sex Ratio

After investigating of the baseline model and completing the sensitivity analysis, the focus of our study now shifts towards an examination of their capacity to modify their sex ratio, which is one of the most important elements that we are interested in throughout the research. This phase of the research aims to delve deeper into the intricate mechanisms that underlie sex ratio plasticity in lampreys.

In particular, we compare our model with that where a species doesn't possess a sex-determination mechanism, thus has relatively even distribution. In other words, $p = 0.5$ is fixed. We shall assume that the context is random similar to the example in Section 4.3.1, and then compare how the distributions for different values change. We generate 100 simulations for each facet of the population's dynamics, as presented below.

5.1 Total Population

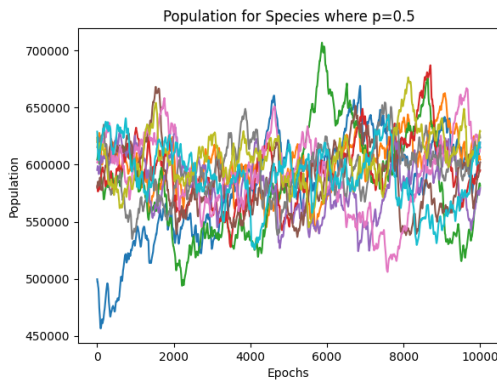


Figure 21: Population of Non-sex-altering Species

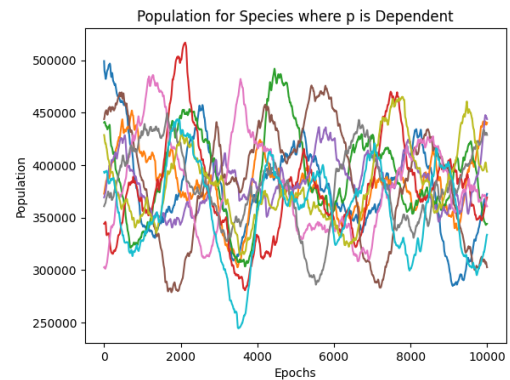


Figure 22: Population of Sex-altering Species

Comparing figures 21 and 22, we see that species possessing a sex-determination mechanism exhibit higher variance around their average population size when compared to species with even sex ratios. In addition, they are lower in absolute population size, indicating a possible disadvantage for sex-altering species. It is important to note that, however, this result may not reflect reality, as we don't consider the parasitic nature of sea lampreys and the magnitude of their effects on other populations within

fisheries [8], a limitation in our model which perceives ecosystems in a simplified framework with standard predator-prey interactions.

5.2 Food Availability

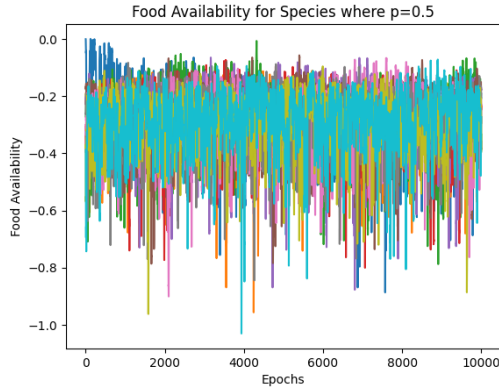


Figure 23: Food Availability for Non-sex-altering Species

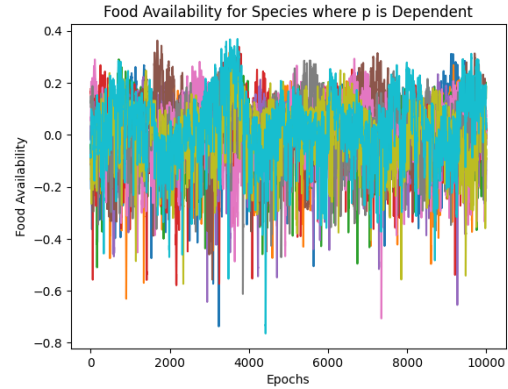


Figure 24: Food Availability for Sex-altering Species

The most prominent advantage for sex-altering species is shown in food availability, where the simulations center around the value 0, while those for non-sex-altering species are centered around -0.3 , meaning that overall, resources for consumption are more abundant relative to the size of species populations. Part of this may be due to the fact that sex-altering species. However, we deduce that the main reason for more abundant food resources for sea lamprey lie in their sex-altering abilities, in that they are more flexible to respond to changes within their environment, and can mitigate to changing resource availability more readily than can species lacking such an ability.

5.3 Population of Predators

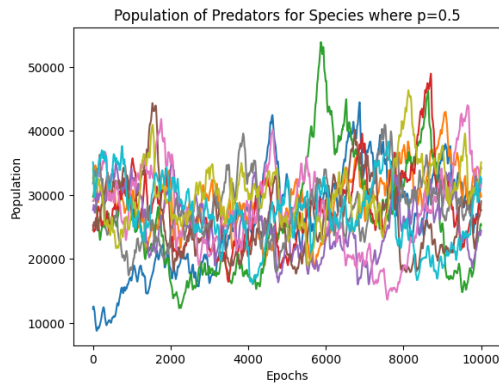


Figure 25: Predator Populations for Non-sex-altering Species

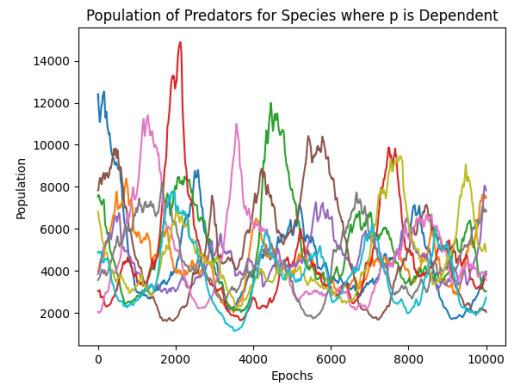


Figure 26: Predator Populations for Sex-altering Species

Comparing figures 25 and 26 we notice a parallel in the differences between the two graphs in predator population and their own species population. The number of predators for sex-altering

species, again, appears to possess higher variability and is comparably lower in mean value. This can be attributed to the construction of our model, and is one of the challenges posed by the problem, in that the complexity of species-interactions within an ecosystem is not exhaustively captured with our characterization of predator-prey dynamics, which is important to consider when associating particular advantages with the ability to alter sex.

5.4 Birth Rate

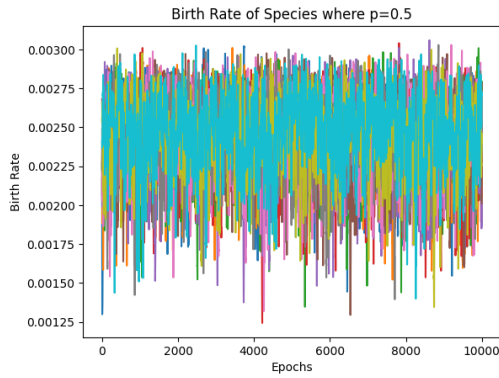


Figure 27: Birth Rate of Non-sex-altering Species

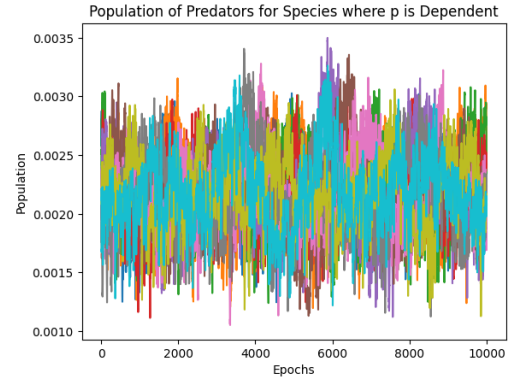


Figure 28: Birth Rate of Sex-altering Species

Based off of figures 27 and 28, we notice that the birth rates of both species are quite similar, where the birth rate of species with even sex ratios seem to be slightly higher than those with a sex-determination mechanism. Conversely, species with a sex-determination mechanism display larger fluctuations in their birth rates. This variability is primarily attributed to their ability to adjust their sex ratio, influencing the birth rate as needed. This adaptability enables them to regulate their birth rate under varying circumstances. In contrast, species with a fixed sex ratio maintain a relatively constant birth rate due to a consistent number of males and females, resulting in a more stable reproduction rate. Consequently, while a lower average birth rate may be considered a disadvantage for species like lampreys, their capacity for variations can confer advantages in other ecological aspects, as observed previously.

5.5 Mortality Rate

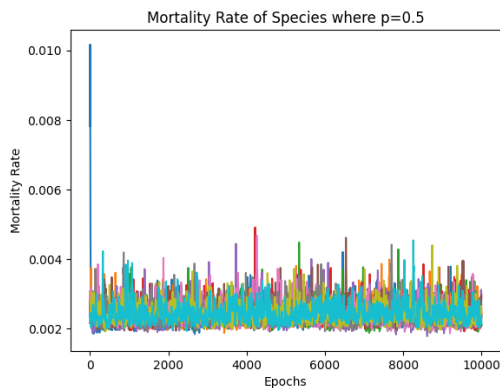


Figure 29: Mortality Rate of Non-sex-altering Species



Figure 30: Mortality Rate of Sex-altering Species

Examining figures 29 and 30, it is evident that species with even sex ratios exhibit, on average, lower mortality rates and variance compared to those with sex-determination mechanisms such as lampreys. According to equation 3.3.5, this suggests that species capable of altering sex have, on average, a larger relative size in their predator population. One proposed explanation is that these species maintain a comparatively lower population than their predators. However, it is crucial to acknowledge that this may not be the sole underlying factor, as our current models do not comprehensively capture all the organisms within an ecosystem.

6 Answers to the Research Questions

Finally, we stand prepared to address the research questions outlined in the initial problem statement. As we near the conclusion of our study, the forthcoming answers are positioned to make significant contributions to the broader comprehension of species with the capacity to alter their sex, along with the broader implications for entire ecosystems.

- (a) What is the impact on the larger ecological system when the population of lampreys can alter its sex ratio?

From the simulations of our model, we can see that the ability to alter sex has allowed lampreys to respond more swiftly to changes in their local conditions. Our results show that food resources are, on average, more available for lampreys compared to non-sex-altering species. Since lampreys play the role of predator to their food sources in ecological systems (as a parasite or otherwise), their flexible nature would inextricably provide more space for other components of the ecosystem which are in relationships of symbiosis / commensalism / mutualism / predation with the lampreys. Thus higher probabilities for abundant resources and stability for other species follows from lampreys' sex-determining mechanisms.

- (b) What are the advantages and disadvantages to the population of lampreys?

As discussed in section 5, our study concludes that the adaptive capacity to manipulate the sex ratio provides lampreys with a significant advantage in locating food resources, distinguishing them from species incapable of such adjustments. Additionally, this adaptive trait appears to contribute to a decreased population of predators. Conversely, lampreys exhibit a relatively lower birth rate and higher mortality rate in comparison to other species, resulting in a diminished overall population.

The observed trend may be attributed, in part, to the incomplete representation of species in our ecosystem data. This incompleteness affects the accuracy of predator-prey relationships, particularly concerning lampreys, which often thrive as parasites and heavily rely on other species. Consequently, a comprehensive understanding of lamprey survival and population dynamics requires accurate data on numerous other species within the ecosystem.

- (c) What is the impact on the stability of the ecosystem given the changes in the sex ratios of lampreys?

Our model suggests that the capacity of lampreys to alter their sex ratios can benefit the stability of ecosystems. This stems from the inherent advantage of enhanced adaptability exhibited by species capable of such sex ratio adjustments. As elaborated in section 5.2, species possessing the ability to modify their sex ratios often possess greater food availability, consequently creating a scenario where there is more room for other species within the ecosystem to forage for their own food resources. This dynamic interplay not only benefits the adaptable species directly but also fosters a more balanced and stable ecological environment, with a potential cascading effect on the overall health and sustainability of the ecosystem.

- (d) Can an ecosystem with variable sex ratios in the lamprey population offer advantages to others in the ecosystem, such as parasites?

In section 5.2, our simulations reveal that species like lampreys consistently exhibit higher average food availability compared to those with a fixed sex ratio. This advantage may benefit other species, including parasites that share the same prey, mainly fish. Yet, it's crucial to note that our model doesn't factor in inter-species competition, potentially affecting the actual significance of this advantage in real-world ecological scenarios.

7 Strengths and Weaknesses

7.1 Strengths

7.1.1 Comprehensive Background Research

Prior to the construction of our model, we conducted a considerable amount of research related to the population dynamics of sea lampreys, and current literature on how they are affected by different environmental factors. Since our model is intended to provide predictive insights into how sea lamprey sex ratios oscillate in response to local conditions, thoroughly gauging the present composition of sea lamprey populations along with their environmental landscapes is integral to evaluate the accuracy of our model results.

7.1.2 Function Construction

In the construction of our model, each function that is included consists of parameters that are determined, based on background and literature research, sufficiently significant to incur changes in the value of interest. Thus, also an ecosystem is all-encompassing, to make predictions that are within a reasonable scope of the realistic circumstance, it is only necessary to include a subset of environmental factors, which is how we proceed with the development of our model.

7.2 Weaknesses

7.2.1 Simplification of Ecosystem Complexity

The most evident of the limitations of our approach roots in the nature of biological interactions, where the inherent inter-connectivity of this problem poses a challenge for us to distinguish relevant variables and their effects on one another. Although we possess strong theoretical support for the

validity of our model, we nevertheless had to make assumptions that somewhat deviate from the conditions within a realistic ecosystem.

Specifically, our model falls short in encapsulating the intricate dynamics of the predator-prey relationship, a crucial aspect that exerts a substantial influence on both the predator population and the overall population of lampreys, a focal point of our in-depth investigation in this paper.

7.2.2 Order of updating

In our model, the variables undergo sequential changes in response to alterations in the ecosystem's environment. This means that the determination of each variable is contingent upon the values of its preceding variables. However, in reality, the dynamics of the ecosystem are more intricate, with various factors influencing each other simultaneously. Our current model, unfortunately, falls short in capturing this concurrent interplay among factors, posing a challenge in accurately representing the complexity inherent in the ecosystem's interactions. Addressing this limitation is crucial for refining the model's realism and enhancing its predictive capabilities.

7.2.3 Data Limitations

Sea lampreys play varying roles in different habitats. As mentioned in the problem statement, they can be parasites or food sources, depending on locality. Thus, to generalize the changes in population dynamics to an arbitrary ecosystem, we cannot treat the relationships between lampreys and their predators/ food sources as strictly parasitic or otherwise. In addition, there is very limited data on the population changes in lampreys within a general ecosystem context, so we lack detailed information on their population sizes, migration patterns, and ecological interactions with their surroundings. This detracts from our ability to assess the degree to which our model is realistic, compared to current landscapes.

8 Results and Discussion

In this paper, we built a system of differential equations in order to model the complex populations of male and female lampreys based off of their ability to change their sex ratios. We then simulated this system over a long period of time and determined its behavior as well as the changes such as the habitat, predators, and food availability. We analyzed the model's sensitivity to different parameters and then compared it to species whose sex ratio is constant. Finally, we determine that the ability for a species to change its sex ratios provides certain advantages such as increasing the food availability which can contribute to the stability of an ecosystem, but also certain disadvantages, such as lower populations and birth rates.

8.1 Further Direction

There are several directions that we can take in order to improve our model. For one, due to the time constraint of the competition, we aren't able to fully fine-tune our model to reflect numbers that align precisely with presently-available data (this is compounded by the fact that data availability is quite scarce for the topic of interest). Thus, we could find more data sources compared to which we can calibrate our model further. Secondly, we can consider different cases for the role of lampreys within ecosystems, where we look at lampreys as parasites and food sources separately, as the main role of their presence in an environment can impact significantly how they interact with other components. Furthermore, we could add complexity to our model through characterizing its interactions with predators/ food sources in a more meticulous manner, i.e. specifying the types of ecological relationships they exhibit with an identified factor in the model. With these changes, our model can be improved to better encompass the complexities present in realistic ecosystems, and

reflect, to a greater extent, the detailed population dynamics and changes which follow from adjusted conditions.

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AI Use Report

- OpenAI ChatGPT (Feb 5, 2024 version, ChatGPT-3.5)

Query1: <Please illustrate the advantages and disadvantages of using logistic regressions to model probabilities>

Output: <Certainly! Logistic regression is a popular statistical method used for modeling binary outcomes, where the dependent variable is categorical with two possible values (e.g., 0 or 1, yes or no). Here are some advantages and disadvantages of using logistic regression to model probabilities:

Advantages:

1. **Simple Interpretation:** - Logistic regression provides coefficients that are easy to interpret. The coefficients represent the change in the log-odds of the outcome for a one-unit change in the predictor variable.
2. **Probabilistic Output:** - Logistic regression models the probability of a binary outcome, producing outputs between 0 and 1. This makes it suitable for tasks where understanding the likelihood of an event is important.
3. **Works well with Small Sample Sizes:** - Logistic regression tends to perform well even with relatively small datasets, making it a good choice when data is limited.
4. **Less Susceptible to Overfitting:** - Compared to more complex models, logistic regression is less prone to overfitting, making it robust when dealing with noisy data.
5. **Efficient Computation:** - The optimization process in logistic regression is well-defined and efficient, allowing for faster training on large datasets compared to some more complex models.

Disadvantages:

1. **Assumes Linearity:** - Logistic regression assumes a linear relationship between the independent variables and the log-odds of the outcome. If the true relationship is highly non-linear, logistic regression might not capture it well.
2. **Doesn't Handle Non-Independence of Observations Well:** - Logistic regression assumes that observations are independent of each other. If there is dependence among observations, it might lead to biased estimates.
3. **Sensitive to Outliers:** - Logistic regression can be sensitive to outliers, which may have a significant impact on the estimated coefficients and predictions.
4. **Limited to Binary Outcomes:** - Logistic regression is designed for binary outcomes. While modifications exist for multiple categories (multinomial logistic regression), it might not be the best choice for problems with more than two outcome classes.
5. **Assumes No Perfect Separation:** - Logistic regression may not perform well when there is perfect separation in the data, where a predictor variable perfectly predicts the outcome. This can lead to convergence issues or infinite parameter estimates.

In summary, logistic regression is a powerful tool for modeling probabilities in binary outcomes, but its appropriateness depends on the specific characteristics of the data and the underlying assumptions of the model. Care should be taken to ensure that these assumptions are met for reliable results.>

- OpenAI ChatGPT (Feb 5, 2024 version, ChatGPT-3.5)

Query2: <Can ecosystems usually be formulated as standard models, encompassing predator-prey interactions, human activities, resource availability, etc., as factors which impact the fluctuation of the populations of interest?>

Output: <Ecosystems are highly complex and dynamic systems, and formulating them as standard models can be challenging due to the multitude of interacting components and the inherent variability in natural systems. While simplified models, such as predator-prey models, competition models, or food web models, can capture certain aspects of ecosystem dynamics, fully encompassing the complexity of real-world ecosystems is often beyond the scope of a single standard model.

Here are some challenges and considerations:

1. **Biodiversity and Complexity:** - Ecosystems often involve a vast array of species with diverse interactions. Modeling this complexity in a comprehensive way can be challenging. Food webs, for example, are intricate networks of species interactions that are difficult to represent accurately in a standard model.
2. **Spatial and Temporal Variation:** - Ecosystems exhibit spatial and temporal variability. Different regions within an ecosystem may have distinct characteristics, and ecological processes can vary over time. Capturing this variability in a standard model is complex and may require sophisticated spatiotemporal modeling techniques.
3. **Human Interactions:** - Human activities, including resource extraction, land use changes, and pollution, play a significant role in shaping ecosystems. Incorporating human impacts into models adds another layer of complexity, and human behavior is often difficult to predict accurately.
4. **Nonlinear Interactions:** - Ecosystem dynamics are often characterized by nonlinear interactions and feedback loops. Standard models may struggle to capture emergent properties and abrupt transitions in ecosystems.
5. **Adaptation and Evolution:** - Species within ecosystems can adapt and evolve over time, influencing population dynamics. Evolutionary processes introduce additional complexity that may not be adequately addressed in standard models.
6. **Stochasticity:** - Natural systems are subject to inherent stochasticity and randomness. Uncertainties in environmental conditions, population sizes, and other factors can make it challenging to predict future states accurately.

Despite these challenges, models are valuable tools for understanding and predicting ecosystem dynamics. Researchers often use a combination of different models and modeling approaches to capture specific aspects of ecosystems. Integrative models, which combine multiple components and processes, and use a variety of modeling techniques, are becoming more common in ecosystem science.

In summary, while standard models can provide insights into specific aspects of ecosystem dynamics, fully encapsulating the complexity of ecosystems often requires a more holistic and integrative approach, considering the multitude of interacting factors and processes.>

- OpenAI ChatGPT (Feb 5, 2024 version, ChatGPT-3.5)

Query3: <What are some of the commonly used models of predator-prey relation?>

Output: <Predator-prey relationships are fundamental components of ecological systems, and several models have been developed to describe and analyze these interactions. Here are some commonly used models of predator-prey relationships:

1. **Lotka-Volterra Model:** - The Lotka-Volterra model is one of the earliest and most well-known predator-prey models. It consists of a pair of coupled differential equations that describe the dynamics of predator and prey populations over time. The model assumes exponential population growth and a simple functional response.
2. **Holling's Disc Equation (Type I Functional Response):** - Named after C. S. Holling, this model describes the rate at which a predator consumes prey as a linear function of prey density.

It assumes that the rate of prey consumption increases linearly with prey abundance until a saturation point is reached.

3. **Holling's Wedge Equation (Type II Functional Response):** - This functional response model assumes that the rate of prey consumption by a predator increases initially but then levels off as prey density continues to increase. It is often considered a more realistic representation of predator feeding behavior than the Type I functional response.
4. **Holling's Type III Functional Response:** - In this model, the rate of prey consumption initially increases slowly, then more rapidly, and finally levels off. It reflects the idea that predators may need time to learn how to efficiently capture and consume certain prey types.
5. **Rosenzweig-MacArthur Model:** - The Rosenzweig-MacArthur model extends the Lotka-Volterra model by incorporating more realistic features such as prey carrying capacity, prey refuge, and predator switching between different prey types. It aims to capture complexities observed in natural ecosystems.
6. **Beddington-DeAngelis Model:** - This model introduces a prey refuge term and a prey carrying capacity, providing a more refined representation of predator-prey dynamics. The refuge term accounts for the fact that not all prey are available for predation.
7. **Ratios-Dependent Predator-Prey Model:** - This model considers the ratios of predator to prey populations, incorporating the idea that the number of predators relative to prey can influence predation rates and population dynamics.
8. **Functional Response with Prey Refuge:** - Some models incorporate the concept of prey refuge, where a portion of the prey population is unavailable to predators. This can represent physical refuges or areas where predators have difficulty accessing prey.

It's important to note that these models are simplifications of real-world ecological interactions, and they make various assumptions that may not always hold in complex ecosystems. Nevertheless, they provide valuable insights into the dynamics of predator-prey relationships and are widely used in ecological research and modeling. Researchers often adapt and extend these models to better capture the specific features of particular ecosystems.>