Carnegie Mellon University
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21-260, Final Exam Practice Test Solutions

1. (5-10 min) Solve the following initial-value problem

$$y' - 2xy = x^3 e^{x^2},\ y(0) = -1$$

Solution:

We first find the integrating factor:

$$\begin{aligned}
&I(x) = e^{\int -2x\,dx} = e^{-x^2} \\
&\implies \frac{d}{dx}(e^{-x^2}y) = x^3 e^{x^2} \cdot e^{-x^2} = x^3 \\
&\implies e^{-x^2}y = \int x^3\,dx \\
&\implies e^{-x^2}y = \frac{1}{4}x^4 + c_1 \\
&\implies y = \frac{1}{4}x^4 e^{x^2} + c_1 e^{x^2}
\end{aligned}$$

Plugging in our initial value, we have that

$$\begin{aligned}
&-1 = c_1 e^0 = c_1 \\
&\implies y = \frac{1}{4}x^4 e^{x^2} - e^{x^2}
\end{aligned}$$

2. (5-10 min) Using the method of undetermined coefficients, solve the following differential equation:

$$y'' - 16y = 2e^{4x}$$

Solution:

We first find the complementary by setting up the auxilirary equation:

$$\begin{aligned} &r^2 - 16 = 0 \\ &\implies r = 4, -4 \end{aligned}$$

which gives us the complementary solution:

$$y_c = c_1 e^{4x} + c_2 e^{-4x}$$

Using the method of undetermined coefficients, we guess that the particular solution has the form Ae^{4x}. However, this has appeared in the complementary solution, so we adjust our guess to Axe^{4x}. We then have that

$$\begin{aligned} &y = Axe^{4x} \\ &y' = 4Axe^{4x} + Ae^{4x} \\ &y'' = 16Axe^{4x} + 8Ae^{4x} \\ &\implies 16Axe^{4x} + 8Ae^{4x} - 16Axe^{4x} = 2e^{4x} \\ &\implies 8Ae^{4x} = 2e^{4x} \\ &\implies A = \frac{1}{4} \\ &\implies y_p = \frac{1}{4}xe^{4x} \end{aligned}$$

which gives us the final solution:

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4}xe^{4x} \end{aligned}$$

3. (15-20 min) Use the Laplace transform to solve the given initial value problem:

$$y'' - 6y' + 9y = t,\ y(0) = 1,\ y'(0) = 1$$

Solution:

$$\begin{aligned}
&\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t\} \\
&\implies \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \frac{1}{s^2} \\
&\implies (s^2Y(s) - s - 1) - 6(sY(s) - 1) + 9Y(s) = \frac{1}{s^2} \\
&\implies (s^2 - 6s + 9)Y(s) - s + 5 = \frac{1}{s^2} \\
&\implies Y(s) = \frac{1}{s^2(s^2 - 6s + 9)} + \frac{s - 5}{s^2 - 6s + 9} \\
&\implies Y(s) = \frac{1}{s^2(s-3)^2} + \frac{s-5}{(s-3)^2}
\end{aligned}$$

Using partial fractions, we have that

$$\begin{aligned}
&\frac{1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2} \\
&\implies 1 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2 \\
&\implies A = \frac{2}{27},\ B = \frac{1}{9},\ C = -\frac{2}{27},\ D = \frac{1}{9}
\end{aligned}$$

$$\begin{aligned}
&\frac{s-5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2} \\
&\implies s - 5 = A(s-3) + B \\
&\implies A = 1,\ B = -2
\end{aligned}$$

Putting them together, we have that

$$\begin{aligned}
&Y(s) = \frac{2}{27s} + \frac{1}{9s^2} - \frac{2}{27(s-3)} + \frac{1}{9(s-3)^2} + \frac{1}{s-3} - \frac{2}{(s-3)^2} \\
&= \frac{2}{27s} + \frac{1}{9s^2} + \frac{25}{27(s-3)} - \frac{17}{9(s-3)^2} \\
&\implies y = \frac{2}{27} + \frac{1}{9}t + \frac{25}{27}e^{3t} - \frac{17}{9}e^{3t}t
\end{aligned}$$

4. (5-10 min) Find the general solution of the given system

$$X' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} X$$

Solution:

We first determine the eigenvalues:

$$\begin{aligned} &det(\begin{bmatrix} 1-\lambda & -8 \\ 1 & -3-\lambda \end{bmatrix}) = (1-\lambda)(-3-\lambda) + 8 \\ &= \lambda^2 + 2\lambda + 5 = 0 \\ &\implies \lambda = \frac{-2 \pm 4i}{2} = -1 \pm 2i \end{aligned}$$

Using $-1+2i$ as our eigenvalue, we obtain our eigenvector:

$$\begin{aligned} &\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\implies v_1 = 2+2i,\ v_2 = 1 \end{aligned}$$

so we have the eigenvector $\begin{bmatrix} 2+2i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Hence, we get that

$$X_1 = e^{-t}[\begin{bmatrix} 2 \\ 1 \end{bmatrix} cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix} sin(2t)]$$

$$X_2 = e^{-t}[\begin{bmatrix} 2 \\ 0 \end{bmatrix} cos(2t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} sin(2t)]$$

which gives us the general solution:

$$X = c_1 e^{-t}[\begin{bmatrix} 2 \\ 1 \end{bmatrix} cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix} sin(2t)] + c_2 e^{-t}[\begin{bmatrix} 2 \\ 0 \end{bmatrix} cos(2t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} sin(2t)]$$

5. (15-20 min) Using variation of parameters, solve the given system

$$X' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} X + \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

Solution:

We first determine the eigenvalues:

$$det(\begin{bmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}) = (-\lambda)(3-\lambda) + 2$$
$$= \lambda^2 - 3\lambda + 2 = 0$$
$$\implies \lambda = 1, 2$$

We then obtain the eigenvectors:
$\lambda = 1$:

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which gives us the complementary solution:

$$X_c = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We then have the fundamental matrix:

$$\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$
$$\implies \Phi^{-1}(t) = \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

Using variation of parameters, we have that:

$$\begin{aligned}
X_p &= \Phi(t) \int \Phi^{-1}(t) F(t)\, dt \\
&= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt \\
&= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix}
\end{aligned}$$

which gives us the final solution:

$$X = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix}$$

6. (20-25 min) Given the following system:

$$X' = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} X + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t}$$

a) Find e^{At} where $A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$.
b) Using your answer for part a, find the particular solution for the system above using matrix exponentiation.

Solution:

We first determine the eigenvalues:

$$det(\begin{bmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{bmatrix}) = (2-\lambda)(6-\lambda)+3$$
$$= \lambda^2 - 8\lambda + 15 = 0$$
$$\implies \lambda = 3, 5$$

We then obtain the eigenvectors:
$\lambda = 3:$

$$\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 5:$

$$\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

This gives us that:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$
$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Using the fact that $e^{At} = Pe^{Dt}P^{-1}$, we have that

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} e^{3t} & e^{5t} \\ e^{3t} & 3e^{5t} \end{bmatrix} \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{bmatrix}$$

b)

$$X_p = e^{At}\int_0^t e^{-As}F(s)\,ds$$
$$= \begin{bmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{bmatrix} \cdot \int_0^t \begin{bmatrix} \frac{3}{2}e^{-3s} - \frac{1}{2}e^{-5s} & -\frac{1}{2}e^{-3s} + \frac{1}{2}e^{-5s} \\ \frac{3}{2}e^{-3s} - \frac{3}{2}e^{-5s} & -\frac{1}{2}e^{-3s} + \frac{3}{2}e^{-5s} \end{bmatrix} \begin{bmatrix} -2e^{2s} \\ e^{2s}\,ds \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{bmatrix} \cdot \int_0^t \begin{bmatrix} -\frac{7}{2}e^{-s} + \frac{3}{2}e^{-3s} \\ -\frac{7}{2}e^{-s} + \frac{9}{2}e^{-3s} \end{bmatrix} ds$$
$$= \begin{bmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{bmatrix} \cdot \begin{bmatrix} \frac{7}{2}e^{-t} - \frac{1}{2}e^{-3t} + 2 \\ \frac{7}{2}e^{-t} - \frac{3}{2}e^{-3t} - 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{2}e^{3t} - \frac{3}{2}e^{5t} + 3e^{2t} \\ \frac{7}{2}e^{3t} - \frac{9}{2}e^{5t} + 2e^{2t} \end{bmatrix}$$

7. (5-10 min) For the following system:

$$x' = 1 - 2xy$$
$$y' = 2xy - y$$

Identify the critical point(s) and classify them as unstable or stable.

Solution:

Setting both $x' = 0$ and $y' = 0$, we have that:

$$1 - 2xy = 0$$

$$2xy - y = 0$$

$$\begin{aligned} & y(2x-1) = 0 \\ \implies & y = 0 \qquad\qquad \text{(impossible from first equation)} \\ \implies & x = \frac{1}{2}, \; y = 1 \end{aligned}$$

We then calculate the jacobian to be:

$$J = \begin{bmatrix} -2y & -2x \\ 2y & 2x-1 \end{bmatrix}$$
$$\implies J(\frac{1}{2}, 1) = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$$

which has determinant 2 and trace -2. This implies that all the eigenvalues have negative real parts, so it is stable.

8. (10-15 min) Find the fourier series of f:

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ 0, & 0 \le x < 1 \end{cases}$$

Solution:

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx \\ &= \int_{-1}^{1} f(x)\,dx \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{p}\int_{-p}^{p} f(x)cos(\frac{n\pi x}{p})\,dx \\ &= \int_{-1}^{1} f(x)cos(n\pi x)\,dx \\ &= \int_{-1}^{0} cos(n\pi x)\,dx \\ &= \frac{1}{n\pi} sin(n\pi x)|_{-1}^{0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{p}\int_{-p}^{p} f(x)sin(\frac{n\pi x}{p})\,dx \\ &= \int_{-1}^{1} f(x)sin(n\pi x)\,dx \\ &= \int_{-1}^{0} sin(n\pi x)\,dx \\ &= -\frac{1}{n\pi} cos(n\pi x)|_{-1}^{0} \\ &= -\frac{1}{n\pi} + \frac{1}{n\pi} cos(-n\pi) \\ &= \frac{1}{n\pi}(-1 + (-1)^n) \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos(\frac{n\pi x}{p}) + b_n sin(\frac{n\pi x}{p})) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n\pi} sin(n\pi x) \end{aligned}$$

9. (5-10 min) Expand the given function in an appropriate cosine or sine series:

$$f(x) = x,\ -\pi < x < \pi$$

Solution:

Notice that x is an odd function, so we shall write it as a sine series:

$$\begin{aligned}
b_n &= \frac{2}{p}\int_0^p f(x) sin(\frac{n\pi}{p}x)\,dx \\
&= \frac{2}{\pi}\int_0^\pi x\, sin(nx)\,dx \\
&= \frac{2}{\pi}[-\frac{1}{n}x\, cos(nx) + \frac{1}{n}\int_0^\pi cos(nx)\,dx] \\
&= \frac{2}{\pi}[-\frac{1}{n}x\, cos(nx) + \frac{1}{n^2} sin(nx)]|_0^\pi \\
&= -\frac{2}{n}cos(n\pi) \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}$$

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{p}x) \\
&= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} sin(nx)
\end{aligned}$$

10. (10-15 min) Use separation of variables to find a solution to:

$$x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$$

Solution:

Let $u(x, y) = X(x)Y(y)$. We then have that

$$\frac{\partial u}{\partial x} = X'(x)Y(y)$$
$$\frac{\partial u}{\partial y} = X(x)Y'(y)$$

so we can rewrite the above expression as

$$xX'(x)Y(y) = yX(x)Y'(y)$$
$$\implies \frac{xX'(x)}{X(x)} = \frac{yY'(y)}{Y(y)} = -\lambda$$

which gives us the 2 ode's:

$$\frac{xX'(x)}{X(x)} = -\lambda$$
$$\implies xX'(x) = -\lambda X(x)$$
$$\implies x\frac{dX}{dx} = -\lambda X$$
$$\implies \int \frac{1}{X}\, dX = \int -\frac{\lambda}{x}\, dx$$
$$\implies ln|X| = -\lambda ln|x| + c_1$$
$$\implies X = c_2 e^{-\lambda ln|x|}$$
$$\implies X(x) = c_2 x^{-\lambda}$$

By symmetry, $Y(y) = c_3 y^{-\lambda}$. Hence, we have that

$$u(x, y) = X(x)Y(y) = C(xy)^{-\lambda}$$

11. (20-25 min) Given the wave equation:

$$u(0,t) = 0$$
$$u(1,t) = 0$$
$$u(x,0) = f(x),\ 0 < x < 1$$
$$\frac{\partial u}{\partial t}|_{t=0} = 2 \cdot f(x)$$

a) Determine the fourier series of $f(x) = \begin{cases} 2, & 0 < x < 0.5 \\ 0, & 0.5 \leq x < 1 \end{cases}$ by using an appropriate extension

b) Using the product form $u(x,t) = \sum_{n=1}^{\infty} sin(n\pi x)(A_n cos(n\pi t) + B_n\, sin(n\pi t))$, determine A_n and B_n to obtain the final solution.

Solution:

a) Setting t to 0, the only thing remaining are sine terms. Hence, we shall extend $f(x)$ as an odd function, so it's defined from the interval $[-1, 1]$:

$$\begin{aligned} b_n &= \frac{2}{p}\int_0^p f(x) sin(\frac{n\pi}{p}x)\, dx \\ &= 2\int_0^1 f(x) sin(n\pi x)\, dx \\ &= 4\int_0^{0.5} sin(n\pi x)\, dx \\ &= -\frac{4}{n\pi} cos(n\pi x)|_0^{0.5} \\ &= -\frac{4}{n\pi} cos(\frac{n\pi}{2}) + \frac{4}{n\pi} \\ &= \frac{4}{n\pi}(1 - cos(\frac{n\pi}{2})) \end{aligned}$$

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{p}x) \\ &= \sum_{n=1}^{\infty} \frac{4}{n\pi}(1 - cos(\frac{n\pi}{2})) sin(n\pi x) \end{aligned}$$

b) Setting $t = 0$, we have that

$$\begin{aligned} & u(x,0) = f(x) \\ \implies & \sum_{n=1}^{\infty} A_n sin(n\pi x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(1 - cos(\frac{n\pi}{2})) sin(n\pi x) \\ \implies & A_n = \frac{4}{n\pi}(1 - cos(\frac{n\pi}{2})) \end{aligned}$$

Taking the partial with respect to t, we have that

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} n\pi sin(n\pi x)(-A_n sin(n\pi t) + B_n cos(n\pi t))$$

$$\implies \frac{\partial u}{\partial t}|_{t=0} = \sum_{n=1}^{\infty} B_n n\pi sin(n\pi x) = 2 \cdot f(x)$$

$$\implies \sum_{n=1}^{\infty} B_n n\pi sin(n\pi x) = \sum_{n=1}^{\infty} \frac{8}{n\pi}(1 - cos(\frac{n\pi}{2}))sin(n\pi x)$$

$$\implies B_n = \frac{8}{n^2\pi^2}(1 - cos(\frac{n\pi}{2}))$$

Putting them together, we have that

$$u(x,t) = \sum_{n=1}^{\infty} sin(n\pi x)[\frac{4}{n\pi}(1 - cos(\frac{n\pi}{2}))cos(n\pi t) + \frac{8}{n^2\pi^2}(1 - cos(\frac{n\pi}{2}))sin(n\pi t)]$$

12. (15-20 min) Determine the solution to the heat equation given that $L = 2$ and

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

You may directly use the formula in the cheat sheet.

Solution:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi}{L}x)\,dx \\ &= \int_0^2 f(x) sin(\frac{n\pi}{2}x)\,dx \\ &= \int_0^1 x\,sin(\frac{n\pi}{2}x)\,dx \\ &= -\frac{2}{n\pi}xcos(\frac{n\pi}{2}x) + \frac{2}{n\pi}\int_0^1 cos(\frac{n\pi}{2}x)\,dx \\ &= -\frac{2}{n\pi}xcos(\frac{n\pi}{2}x) + \frac{4}{n^2\pi^2}sin(\frac{n\pi}{2}x)|_0^1 \\ &= -\frac{2}{n\pi}cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2}sin(\frac{n\pi}{2}) \end{aligned}$$

where we have that:

$$n\%4 = 0: \; A_n = -\frac{2}{n\pi}$$

$$n\%4 = 1: \; A_n = \frac{4}{n^2\pi^2}$$

$$n\%4 = 2: \; A_n = \frac{2}{n\pi}$$

$$n\%4 = 3: \; A_n = -\frac{4}{n^2\pi^2}$$

So our final answer is

$$u(x,t) = \sum_{n=1}^{\infty}(-\frac{2}{n\pi}cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2}sin(\frac{n\pi}{2}))sin(\frac{n\pi}{2}x)exp(\frac{-kn^2\pi^2}{L^2}t)$$

(If you simply write A_n as a piecewise and just write out the original explicit solution, that is fine as well).