21-270 Problem Sessions Compiled

Jerick Shi

Contents

1 Problem Session 1

Some Pieces of Advice

1. **Don't overthink the problems:** If you want to succeed in this course, stop overthinking some of the problems. We give you all the information you need. If you have 5 stocks and 10 call options, that is the value of the portfolio. Don't think about buying it and somehow having a negative value. Every asset is a mathematical object, nothing more, nothing less.

2. **Don't just remember formulas:** This will apply later in the course, but you should know how every formula is derived. Most of the formulas we give will be built on some sort of replicating portfolio, but on the exam we can easily change one thing, causing the formula to be invalid.

3. **Network with the TA's:** Most people in the review session are considering a career in quant/minoring in computational finance. Most of the other TAs are BSCF majors, all of us have some additional major/minor in math/CS/stats. Some of the TAs have had some internships in some REALLY impressive firms, some of the TAs are quant club board. Talk to the TAs to know what it is like to be a BSCF major / minor, advice on recruiting, or what you could do now. Everyone has a different opinion on the same topic, so it is good to network with many different people early!

Basic Review

A **stock** is an asset that changes value over time. You do NOT know how it changes. All you know is the initial price of the stock today.

When you put money in the **bank**, it grows with **interest rate** r. If you put \$100 in the bank, you will have $\$100(1+r)$ in one year, and $\$100(1+r)^2$ in 2 years etc.

A **call option** has a payout $\max(S_t - K, 0) = (S_t - K)^+$, and a put option has a payout $\max(K - S_t, 0) = (K - S_t)^+$. You do NOT know how to calculate the price of an option right now. We will give you the price explicitly.

A **forward contract** is a promise to buy a stock at a certain price (the forward price $\mathcal{F}$) in the future. When we mean that it costs nothing to enter, we mean that you are just telling Jerick that "hey sign this paper such that I give you money and you give me a stock". No one is paying anything right now. We will later show how to calculate the forward price.

If we **long** an asset with value X, the value is $+X$. If we **short** the asset, the value is $-X$.

Example 1 (Exercise 1.11)

An investor holds a put option, buys one share of stock, and sells short a call option with the same strike price K. Sketch the graph of X_T as a function of S_T.

If you are ever asked to find/sketch the payout, ALWAYS write a formula for the terminal value. In this case, a put option has payout $(K - S_T)^+$, the share has payout S_T, and the call option has payout $(S_T - K)^+$. Together, we have the terminal value to be

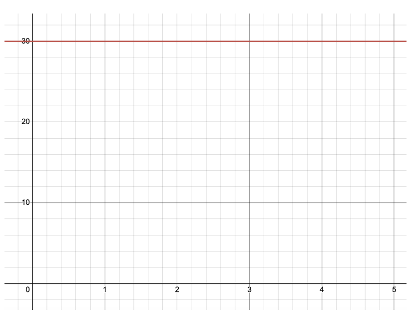
$$X_T = (K - S_T)^+ + S_T - (S_T - K)^+$$

We then do casing to see how the value changes over time. In this case, we would see if $S_T < K$ or $S_T \geq K$:

- $S_T < K$: $K - S_T + S_T - 0 = K$

- $S_T \geq K$: $0 + S_T - (S_T - K) = K$

What does this mean, this means that regardless of how the stock changes, the payout will always be K! Hence, the graph will literally just look like a straight line through K:



Some of y'all might have the intuition "does that mean the call and option prices might be somehow related?" Yes, they are! We will see later in future weeks why that is the case!

Replication

Perhaps the most important statement 21-270: **The price of an asset is the exact same as the price of the replicating portfolio.**

Here is a simple example. Let's say apples are worth \$2 and oranges are worth \$1. Elmer is planning to sell 10 apples and 5 oranges in a bag. What should the price be?

Clearly, we can simply add them. $\$2 \cdot 10 + \$1 \cdot 5 = \$25$. What happens if he sets the price as \$24? Well, I can buy the bag from him, and then sell the apples and oranges for their market price, making a profit of \$1. If I buy 1000 bags, then I make a profit of \$1000. What happens if he sets the price as \$26? Well, we can buy the apples and oranges separately, then sell them to Elmer, still making a profit.

This introduces the idea of **arbitrage**, which in simple terms is basically making money out of nothing, or riskless profit. If there is no arbitrage, then there is a unique price. That is exactly how we will price things in this course. We will assume that there are no arbitrage opportunities and then get a unique price by using the values of the individual components, or, in this case, the replicating portfolio.

Example 2 (Exercise 1.16 modified)

Assume there's a stock with $S_0 = \$98$ and that the bank has interest rate $r = 0.06$. Let's say that you sign a contract such that you receive a stock by paying \$90 in a year. What is the replicating portfolio and what is the value of this contract?

In one year, we will receive one stock, and pay (or owe) \$90. How do we replicate that? Let us first think about the stock. If we buy the stock today, then we will have the stock in one year. Hence, we mimic the act of receiving the stock in a year by buying it today.

What about owing the \$90? Well, if we owe the bank $\$\frac{90}{1.06}$ today, that debt will grow to $\$\frac{90}{1.06} \cdot 1.06 = \$90$ in one year. Hence, this is how we mimic the future debt.

So what is the replicating portfolio? We buy one stock and borrow $\$\frac{90}{1.06}$ today. This will mimic receiving one stock and owing \$90 in one year.

Now all we need to do is calculate the price of this replicating portfolio. We know the initial price of the stock, so we have

$$\$98 - \$\frac{90}{1.06} \approx \$13.09$$

What does this mean? This means that for the contract to be 'fair', you should give me \$13.09 to convince me to sign this contract. Notice that this is no longer a forward contract, since it costs money to enter it.

Let us modify the question. Same stock price and interest rate, but now find the forward price $\mathcal{F}$ such that the forward contract is initially worth nothing.

Similarly, we can replicate the portfolio by initially buying a stock and then borrowing $\$\frac{\mathcal{F}}{1.06}$ from the bank. Since this value has to equal 0, we have that

$$\$98 - \$\frac{\mathcal{F}}{1.06} = 0 \implies \mathcal{F} = \$103.88$$

With this forward price, the contract costs nothing to enter, and no one is exploiting anyone.

Notice that we didn't need to know how the stock changes over time. This is the key idea of pricing: We can price objects using only the price today.

Example 3

Assume an interest rate of $r = 0.1$. Amelia pays you \$100 in year 1, \$100 in year 2, and so on until year 10, where she pays an additional \$1000, so she pays \$1100 in total in the 10th year. What is the replicating portfolio, and what is the price of that portfolio?

One nice thing about the replicating portfolio strategy is that we can treat each payout separately and then add them together. In order to replicate the payment in a year, we borrow $\$\frac{100}{1.1}$ from the bank today. We then replicate the payment in 2 years, where we borrow $\$\frac{100}{1.1^2}$ today. This keeps going where the 10th year, we have $\$\frac{1100}{1.1^{10}}$. Together, the value today should be

$$X_0 = \sum_{i=1}^{10} \frac{100}{1.1^i} + \frac{1000}{1.1^{10}}$$

Example 4

Assume an interest rate of r, and that stocks cost S_0 today. Assume that every year, you pay \$20 for a stock for 3 years. What is the replicating portfolio, and what is the price of that portfolio?

Similar to before, we can replicate the stocks by initially buying 3 stocks, since that will replicate receiving one stock in each year. For the payments, it will just be a summation, similar to what we saw in the last example:

$$X_0 = 3 \cdot S_0 - \frac{20}{1+r} - \frac{20}{(1+r)^2} - \frac{20}{(1+r)^3}$$

Takeaways

1. Replication is key. You will calculate the prices by finding the replicating portfolio.
2. Come to OH to ask for help/network.
3. These problem sessions will be recorded, but you should still try to come in person to have the post-problem session discussions :)

2 Problem Session 2

Welcome back to the second problem session. This time, we covered more about replication and tied that to other ideas in arbitrage strategies and arbitrage-free pricing.

Example 1

At time 1, you receive S_1^1 and pay K_1. At time 2, you receive K_2 and give S_2^1. At time 3, you receive S_3^2 and pay K_3. At time 4, you receive K_4 and pay S_4^2. Assume an interest rate of r. What is the price of the replicating portfolio.

The main goal of this problem is to get used to this new notation. S_0^1 means the price at time 0 of stock 1, while S_0^2 means the time-0 price of stock 2. We will be using this notation when we are dealing with different stocks.

Hopefully everyone is used to creating these replicating strategies. For the time 1 payments, we buy a stock and borrow money from the bank. For the time 2 payments, we short a stock and put money in the bank, etc. Together, we have:

$$X_0 = S_0^1 - \frac{K_1}{1+r} + \frac{K_2}{(1+r)^2} - S_0^1 + S_0^2 - \frac{K_3}{(1+r)^3} + \frac{K_4}{(1+r)^4} - S_0^1$$

which if we wanted to simplify the stock prices canceling out, we have

$$X_0 = -\frac{K_1}{1+r} + \frac{K_2}{(1+r)^2} - \frac{K_3}{(1+r)^3} + \frac{K_4}{(1+r)^4}$$

Example 2

Assume an interest rate of r, How do we replicate earning S_1 at $t = 2$? In other words, how do you earn the stock price at $t = 1$ at time $t = 2$? For example, if the stock price was \$20 today, \$80 at $t = 1$, and \$40 at $t = 2$, we should be paid \$80 at $t = 2$.

It might seem hard to start, but here is a hint: You only know S_0 and r at $t = 0$.

Well, we can buy one stock today and then hold it until $t = 1$, becoming S_1. How do we keep that information until $t = 2$? We could just sell the stock then and then put the money in the bank! That will give us $S_1(1+r)$, so we just need to scale it.

So, here is the strategy: Buy $\frac{1}{1+r}$ of a stock today so that the value is $\frac{S_0}{1+r}$, hold it until $t = 1$ so that the value is $\frac{S_1}{1+r}$, exchange the stock into money and put it in the bank, which then grows to S_1.

Arbitrage + Arbitrage-free Pricing

Here is the definition of an **arbitrage opportunity**:

1. $X_0 = 0$
2. $X_T \geq 0$
3. $X_T > 0$ with positive probability

More officially:

1. $X_0 = 0$
2. $P(X_T \geq 0) = 1$
3. $P(X_T > 0) > 0$

But the logic is still the same: We start off with nothing, and then make risk-less profit. This is different from simply putting money in the bank, as in this case $X_0 \neq 0$. Though you will prove in this homework that if you do better than the bank, that is an arbitrage opportunity.

If the market is arbitrage-free, then there are no arbitrage opportunities. Furthermore, there exists a unique price for every asset, the arbitrage-free price, since if the asset is not that price, we can find an arbitrage opportunity.

Example 3 (Modification of Exercise 1.19)

Consider a stock with $S_0 = \$16$, and an interest rate $r = 0.25$. At $t = 1$, the stock rises to $S_1 = \$32$ with $\frac{2}{3}$ probability and drops to $S_1 = \$8$ with $\frac{1}{3}$ probability. What is the arbitrage-free price $\mathcal{F}$ of the forward contract, and how do we create arbitrage when it is not that price?

The first question should be trivial from last week. However much money we borrow grows into the forward price, so $S_0 \cdot (1+r) = \$16 \cdot 1.25 = \$20$.

Here, I want to highlight one important fact: For the first 70% of this course, **we will not be dealing with real life probabilities**. The reason being when we replicate portfolios, we only care about the outcomes, we don't care how likely they are to happen. An easy way to think about this is that if the underlying asset has some probability of going up or down, the replicating strategy follows that asset, so the probabilities do not matter. Notice in the problem that we did not use any of the probabilities for our answer. Even if we changed them, the price would not change. This concept will be much more important later in the semester when we deal with risk-neutral probabilities.

Now, let's see what happens when the forward price isn't \$20, maybe \$10. Let's buy the forward contract, short a stock, and then put \$16 in the bank. At $t = 1$, the payout becomes

$$(S_1 - \$10) - S_1 + \$16 \cdot (1.25) = \$10 > \$0$$

Notice what we did here. Our initial portfolio, since the forward contract costs nothing, is worth zero. Then we showed that the payout, regardless of what S_1 is, will always be a positive profit. This is how we will be creating arbitrage strategies.

This argument applies to all prices where $\mathcal{F} < \$20$. Now the question is, how do we prove that $\mathcal{F} > \$20$ provides an arbitrage strategy as well? Hint: Just flip everything!

Often, when you are trying to make arbitrage strategies, you might accidentally show that you will always lose money. Notice that if you just flip your strategy, long to short, or vice versa, that just becomes always making money!

Example 4 (Modification of Exercise 1.22)

Consider 2 stocks where $S_0^1 = S_0^2 = \$20$. There are 3 events that might occur in the future, $\omega_1, \omega_2, \omega_3$. If ω_1 happens, $S_1^1 = \$24, S_1^2 = \$24$. If ω_2 happens, $S_1^1 = \$18, S_1^2 = \$20$. If ω_3 happens, $S_1^1 = \$16, S_1^2 = \$12$. Each event occurs with equal probability.

Consider the security that pays the following:

$$V_1(\omega_i) = \max\{S_1^1(\omega_i), S_2^2(\omega_i)\}$$

In other words, the maximum value of the 2 stocks. Explain why we know that $\$20 < V_0 < \$\frac{24}{1.1}$ without explicitly calculating the price.

The main idea is to show that if we are outside of this range, there exists an arbitrage opportunity.

Consider proving $V_0 \geq \$\frac{24}{1.1}$. If we sell short this security, and use that money to put in the bank, which causes our initial portfolio to have value 0, the bank will give us a payment $\geq \$24$. In order to pay off the debt of this security, notice that the maximum possible value is \$24. Hence, since we will never lose money and there are cases where we make a positive profit, we created an arbitrage strategy.

If this is difficult to follow, consider setting V_0 to something very large, like 100. Then convince yourself that with the above strategy you will always make money.

Now consider the case where $V_0 \leq \$20$. Long the security and short S^1 or S^2, and then put any extra money in the bank. Since the security always pays the max of the 2 stocks, we will never lose money from the security subtracting a stock, and we will make a bit of extra money from the bank.

How to price with multicases

Consider a universe that is governed by a single coin flip. For simplicity sake, let $r = 0$. Let's say there's a stock currently worth \$20 such that if the coin flips heads at $t = 1$, it's worth \$30. Otherwise, it's \$10. How do we replicate the following payouts and what is the price of the replicating portfolio?

Case 1: \$60 if heads, \$20 if tails.
Just buy 2 stocks. $2 \cdot \$20 = \$40$.

Case 2: \$70 if heads, \$30 if tails.
Buy 2 stocks, but also put \$10 in the bank. $\$40 + \$10 = \$50$.

Case 3: \$20 if heads, \$40 if tails.
Put \$50 in the bank, and short one stock. $\$50 - \$20 = \$30$

Side Note: I would like to continue emphasizing that in all these calculations, we never used real world probabilities.

It seems that whatever 2 payouts we have, we can always buy a certain amount of stocks and put a certain amount of money in the bank to replicate it. Why? Let us see what happens.

Say we buy α stocks and put β in the bank. The initial portfolio would then be worth $\alpha S_0 + \beta$, which at time 1, becomes $\alpha S_1 + \beta(1+r)$. If we split into cases, we have

$$\alpha S_1(H) + \beta(1+r) = V_1(H)$$

$$\alpha S_1(T) + \beta(1+r) = V_1(T)$$

$r, S_1(H), S_1(T)$ were given, what we were doing is setting different $V_1(H)$ and $V_1(T)$, so what we really are doing is solving a system of equations for α and β. With these values, we know how much stock to hold and how much money to put in the bank; hence, we know the value of the replicating portfolio, hence the price of the security.

Example 5 (Exercise 1.24)

Consider a stock where $S_0 = \$44$, and the 3 possibilities are $S_1(\omega_1) = \$80$, $S_1(\omega_2) = \$60$, $S_1(\omega_3) = \$40$. There is also a put option with that underlying stock and strike price $K = \$48$ whose price is $P_0 = \$2.40$. Let the interest rate be $r = 0.25$. Consider a call option with strike price $K = \$64$. Find a replicating strategy for the call option and determine its price.

First, to make things clearer, let's explicitly calculate the cases for the put and call options:
$P_1(\omega_1) = \$0$, $P_1(\omega_2) = \$0$, $P_1(\omega_3) = \$8$
$C_1(\omega_1) = \$16$, $C_1(\omega_2) = \$0$, $C_1(\omega_3) = \$0$

Now we do something similar to before, we buy α stocks, β put options, and put γ in the bank. This gives us the system of equations

$$80\alpha + 0 + 1.25\gamma = 16$$

$$60\alpha + 0 + 1.25\gamma = 0$$

$$40\alpha + 8\beta + 1.25\gamma = 0$$

Algebra should give us that $\alpha = 0.8$, $\beta = 2$, and $\gamma = -38.4$. This means that replicating strategy should be buying 0.8 stocks, 2 put options, and getting a loan of $-\$38.4$ from the bank. This gives us the initial price of

$$0.8 \cdot \$44 + 2 \cdot \$2.40 - \$38.4 = \$1.60$$

Which is a small price similar to a call option, which we expect.

Conclusion

In this problem session, we drilled the idea that replication is pricing. Every price we got is by creating a portfolio that replicates the payouts. If we do not hit that exact price, arbitrage opportunities occur, where we can make money out of nothing.

21-270 Problem Session 3

And here we are with the third problem session! Today, we focused more on interest rates and then did some exercises to prepare for the first exam!

3 Problem Session 3

Example 1

Consider the following problem: An asset V has 3 possible payouts, $V_T = \$100, \$200, \$300$, each occurring with equal probability. Let the interest rate be r.

A student makes the following argument. The average payout is given by $\frac{1}{3}(\$100 + \$200 + \$300)$, and since the money in the bank grows at a rate of r, the final price should be

$$V_0 = \frac{1}{1+r} \cdot \frac{1}{3}(\$100 + \$200 + \$300)$$

Is this a good argument? No! This is simply calculating how much we would earn on average, which is not what we do for arbitrage-free pricing. The derivatives would depend on some underlying asset and the main idea is that the price should stay the same regardless of how the probabilities change, since the price comes from some combination of the stocks and the bank.

Interest Rates

Recall what we were doing with compound interest rates. If we add A amount of money in T years, we would have $A(1+r)^T$. For semiannual compounding, the money would grow at $A(1+\frac{r}{2})^{2T}$ (since we are paying twice the time), and for quarterly, it would be $A(1+\frac{r}{4})^{4T}$.

Here, we introduce the idea of **effective spot rates** $R_*(T)$. The $_*$ basically means that we are setting this value TODAY, aka it is known TODAY. The effect of the spot rate basically says that at time T, A would grow to $A(1+R_*(T))^T$, regardless of what type of compound we are using. I like to think about it as a "simple annual rate", which means that the formula for this is quite simple.

What we had before in compounding is called the **nominal rate**. If we have something like $r[4]_*(3)$, then A would grow to $A(1+\frac{r[4]_*(3)}{4})^{12}$ at $T = 3$. This is where I want to highlight an important point: Whenever a rate is given for a period of time T, use it ONLY for that period of time T. You should not use $r[4]_*(3)$ to calculate something at $T = 7$, only $T = 3$.

We also introduce the concept of **discount factors**. Basically, one dollar at time T costs $\frac{1}{(1+R_*(T))^T} = d(T)$ today. If we have multiply payouts, we can write something like $100d(1) + 200d(4) + 300d(5.5)$, where we do not see any of the interest rates/compounding. I want to highlight that we changed nothing about the math, we just made it look cleaner. Typically, we would simplify the math until we only have discount factors and then plug in the actual fractions of interest rates.

Forward Interest Rates

$R^{for}_{\tau,\eta,T}$ is called the **effective forward interest rate**, or, in simple terms, the effect interest rate set at τ for money from η to T. In other words, 1 dollar at time η should be worth $(1+R^{for}_{\tau,\eta,T})^{T-\eta}$ at time T. How do we replicate it?

In lecture, we gave a very simple idea, consider longing a dollar at $t = \eta$ and shorting $(1 + R^{for}_{\tau,\eta,T})^{T-\eta}$ at $t = T$. The time-zero price should be 0, or else there would be arbitrage. Discounting them, we have that

$$d(\eta) = d(T) \cdot (1 + R^{for}_{\tau,\eta,T})^{T-\eta}$$

in other words

$$\frac{1}{(1 + R_*(\eta))^\eta} = \frac{(1 + R^{for}_{\tau,\eta,T})^{T-\eta}}{(1 + R_*(T))^T}$$

which using algebra, the notes gives us that

$$R^{for}_{\tau,\eta,T} = (\frac{(1 + R_*(T))^T}{1 + R_*(\eta))^\eta})^{\frac{1}{T-\eta}} - 1$$

Feel free to write this formula in your cheat sheet, but make sure you KNOW how we got it (the replication process etc).

How to survive 21-270 Exams

You have 50 minutes to work on 4 problems. That is on average 10 minutes per problem, and 10 extra minutes to start over one problem when you realize that you did it completely wrong. That's not a lot of time. From my experience, people who bomb a problem fall into one of 2 categories.

1. They run out of time.
2. They go in a random direction that makes 0 sense.

They are also kind of related: If you waste all your time on a problem since you are taking the round-about way, you will not have time to do the other problems. In addition, you should not be spending that much time thinking about how to solve the problem. In the second half of problem session, we went through some examples on how we should be able to identify a problem and the overarching process in under a minute.

Example 2.1

Let's say the problem looks like this: Suppose $X_T - Y_t \geq 0$ always, $X_t - Y_t > 0$ almost surely, prove that XXXXX. What do you think the problem is asking, and how would you approach it?

You should realize that this is some sort of arbitrage proof problem. The final sentence will either make you find an arbitrage strategy, prove that there is no arbitrage, or show that there is no arbitrage iff something exists.

The approach should be the same: Create an explicit arbitrage strategy with initial value 0, show it satisfies the final conditions, and that either directly proves arbitrage, or is a contrapositive of an iff statement. If you follow this process correctly, even if you might not get exactly what you want, I can guarantee that you can get at least around 10 points.

Example 2.2

There is a stock price with 3 possible values. There is a strategy X with 3 possible payouts, A, \$1000, \$1000. What are the possible questions and what are their approaches?

Show that A or X_0 has to be within some bound such that there is no arbitrage? Assume the opposite and explicitly create an arbitrage strategy!

Explicitly calculate X_0? Use the systems-of-equations strategy with replication!

Example 2.3

You have 10 put options, short 20 call options, long 10 stocks, etc. etc.

We could ask the payout, plot the final payout, find the initial price of the portfolio, YOU SHOULD KNOW HOW TO DO THESE.

Using the Practice Problems

Take 30s to look at problem 4 on the practice problem. What do you see? You see part a, and you immediately think: I should probably construct some replicating portfolio. You would probably then need to calculate the possible values of V_1. The 1.1 should be a hint that you need the bank, and the 0 should be quite intuitive. You then look at part b. You know you should set up a system of equations: Use the 2 stocks and the bank, that's 3 variables!

Take 30 seconds to look at problem 5. You see a foreign currency, so you know it is related to exchange rates. You see a forward contract to trade a stock for money. You immediately think that you should hold one stock and have a discounted money. Since there are foreign currencies, you might need to exchange them to get the final answer.

What about problem 8? You immediately know this is an arbitrage proof strategy. You see an iff, so you take the contrapositive. You know that you will probably need to create an arbitrage strategy with an Arb pair and an Arb pair using the fact that a portfolio can sporadically make money out of nothing.

Problem 11? You see part a and the many stocks and you know it is a system of equations. Without even finishing part a, you see part b and you know that you can use the stock price to directly calculate the forward price. In part c, you know that if it does not match with the forward price, there is some arbitrage strategy.

Notice how in just a few words, I basically summarized the entirety of how to do each problem. For most people, going down the right path should give you at least $10-15$ points, and even if you only make a calculation error, that is -3 MAXIMUM. You should have as much time as possible actually doing the calculation and not wasting time thinking about the direction, especially the wrong direction.

How to Prepare

There are 2 types of students taking this class: You are either taking this as a free elective and everything so far is quite trivial (Type A), or you are spending a lot of time on homework and the concepts are taking a while to kick in (Type B).

If you are a Type A student, go through all the review problems, using that 30s intuition to think about the entire problem. If you feel like everything makes sense, go to the next problem. If something is stuck, solve it.

If you are a Type B student, start off by going over the notes, making sure you did not miss anything. Then go through the homework, especially the solutions. Finally, work on practice problems, especially those that seem challenging. Time yourself. If calculation is a time crunch, keep practicing them, so that you are used to using the same formula but just different numbers.

Regardless of how you are studying, I wish you the best of luck! You got this!

4 Problem Session 4

Welcome back! Hope the midterm was not too hectic! A bit less attendance today, which is fine, but even though the sessions are recorded, I still encourage people to come! It will make my life a lot easier, and you get to ask whatever questions you want!

Fixed Income Instruments

When we say **fixed income**, we mean assets where we know how much money we are earning at a specific time, so we will not be dealing with stocks. We go over what the payments are as well as their time-0 prices as a quick review:

	Payout	Time-0 Price
ZCB	F at time T	$F \cdot d(T)$
Annuity	A until time T	$\sum_{i=1}^{T} A \cdot d(i)$
CB	$F \cdot \frac{q[m]}{m}$ until time T + payment F	$\sum_{i=1}^{mT} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m}) + F \cdot d(T)$

I claim that given any 2 of these assets, you can price the other one. Let's give an example:

- A ZCB has price P_1 pays $F = \$100$ at $T = 10$
- An annuity has price P_2 pays \$10 each year until $T = 10$
- A CB has price P_3 that pays \$5 each year until $T = 10$ and pays $F = \$200$

Let's say you know P_1 and P_2, how would you price P_3? Well, we can replicate the payments. For the annual payments, hold only half the annuity. Then, just hold 2 of the ZCB. That portfolio has the exact same payouts as the CB, so using our knowledge of replication, they have the exact price! Hence, $P_3 = 2P_1 + \frac{1}{2}P_2$. Hopefully, this is convincing enough that given the prices of a ZCB and Annuity, you can price any coupon bond.

Now let's say you know P_1 and P_3, can you price annuity? Yes! Long the CB, short enough ZCB such that the final payout cancels out, so all you get are annual payments. You can then scale that payout to get any annuity.

Finally, let's say you know P_2 and P_3. A similar idea occurs: Long the CB and short the right amount of annuity, so the annual payments cancel out. All that's left is the final payment, which you can scale.

Effective Internal Rate of Return

Recall that when we had r, I said that you can think of it as the 'actual' interest rate that changes in a year, and that $R_*(T)$ and be thought of as the 'averaged out' yearly interest rate where you can directly apply $(1+R_*(T))^T$. The **effective interval rate of return** R_I can be thought of as the 'average' of the spot rates. I think the best way to explain this is by looking at the definition. If we have a lot of payments, the price can be given as

$$P = \sum_{i=1}^{N} \frac{F_i}{(1+R_*(i))^{T_i}} = \sum_{i=1}^{N} \frac{F_i}{(1+R_I)^{T_i}}$$

Conceptually, it can be thought of as an average of all the different spot rates for a given security. Mathematically, it can be thought of as a constant interest rate. It is important to note that every asset has its own R_I.

One reason why this is so powerful is that it just makes calculations easier. If we know the R_I of an annuity, then we know that

$$P = \sum_{i=1}^{T} \frac{A}{(1+R_I)^i} = A \cdot \sum_{i=1}^{T} \frac{1}{(1+R_I)^i} = A \cdot \sum_{i=1}^{T} \lambda^i$$

where we let $\lambda = \frac{1}{1+R_I}$. Then, we use the geometric series shortcut:

$$\sum_{i=1}^{k} \lambda^i = \lambda \cdot \frac{1-\lambda^k}{1-\lambda}$$

To make the above expression look much nicer. Hence, given an R_I, we can calculate λ, and directly calculate the price!

We also talk about the **nominal IRR**, which is just a formula:

$$(1+\frac{r_I[m]}{m})^m = 1+R_I$$

In other words, take our 'averaged' yearly interest rate, and decompose it into m periods in a year.

Example 1

Say for a portfolio V, we have $A < V_T < B$ always, where A and B are real numbers. Prove $\frac{A}{1+r} < V_0 < \frac{B}{1+r}$.

Here's an incorrect argument: Since V_T has to be between A and B, we know that V_T has to be $\alpha A + (1-\alpha)B$ for some $\alpha \in [0,1]$, in other words, V has to be a linear combination of A and B, so the initial price should be discounted just between them.

If you say that V has to be this linear combination, you basically say that V_T has to be constant. You are not arguing that this is true for every possible V_T, you are arguing for a specific subset of V_T's where the payout is constant. Don't make this mistake in future hws/exams. Know what we mean when we say for all/there exists, why we use proof by contradiction, when counterexamples are valid, etc. If necessary, I might create an introduction proof document to make sure everyone is up to speed.

(The right solution, as most people will expect, is to assume the contradiction and create an arbitrage strategy.)

Example 2

Consider these 2 different assets:

- A coupon bond with face value $F = \$10,000$ with 2 payments per month $q_1[2] = 0.05$, maturity $T = 10$, with an initial price \$9179.10
- An annuity that pays \$500 twice a year, maturity $T = 10$, and $R_I = 0.05$.

Part A: What is the arbitrage-free price of the annuity?

This is where we use the fact that R_I makes calculations a lot easier:

$$P = \sum_{i=1}^{20} \frac{500}{(1+R_I)^{i/2}} = 500 \cdot \sum_{i=1}^{20} \lambda^i$$

where we use the same substitution. Using the geometric series, we get that

$$P = 500\lambda \cdot \frac{1-\lambda^{20}}{1-\lambda}$$

and by calculating $\lambda = 0.9759$ and substituting, we should get $P = \$7817.08$.

Part Pre-b: Can we calculate the price of any ZCB now? Well, we have the prices of a coupon bond and an annuity, so yes!

Part B: Calculate $d(10)$

This is where we use the intuition above. Since we can calculate any ZCB, if we know that the price of a ZCB that pays Y is X, our formula tells us that $X = Y \cdot d(10)$, so we can directly solve for $d(10)$.

In our case, the CB pays \$250 twice per year, so if we long the CB and shorting half the annuity, we are left with a ZCB with $F = 10,000$ and price $P^{CB} - \frac{1}{2}P^A = 9179.10 - \frac{1}{2}7817.08 = 5270.56$

Therefore,

$$d(10) = \frac{5270.56}{10000} = 0.527$$

Part C: What is the arbitrage-free price of a coupon bond with face value $\$5000$, maturity $T = 10$, and coupon rate 0.06, making coupon payments twice per year?

I will leave this exercise to the reader, but the main idea is that you can directly replicate it either with the annuity and knowing $d(10)$, or a combination of the original CB and some $d(10)$.

Example 3

Consider these 2 different assets:

- An annuity paying \$500 payments once per year for 10 years with an internal rate of return of $R^A_{IRR} = 0.03$.
- A ZCB with $F = \$10,000$ and maturity 10 years with an internal rate of return of $R^B_{IRR} = 0.04$.

Part A: Find the arbitrage-free price of a CB with $F = \$1000$ and maturity payments making coupon payments at rate $q[1] = 0.03$

Let's first think on a higher level on what we wanna do. We probably want to have some combination of the annuity and the ZCB to get the price of the CB, but we don't have the prices of those assets. We do that for the R_I's, so we can calculate the prices first and then do the replication!

We first work on the ZCB:

$$P^{ZCB} = \frac{10,000}{1.04^{10}} = 6755.64$$

For the annuity, the process is similar as before:

$$\begin{aligned}
P^A &= \sum_{i=1}^{10} \frac{500}{(1 + R_{IRR}^A)^i} \\
&= 500 \cdot \sum_{i=1}^{10} \frac{1}{(1 + R_{IRR}^A)^i} \\
&= 500 \cdot \sum_{i=1}^{10} \lambda^i \\
&= 500\lambda \cdot \frac{1 - \lambda^{10}}{1 - \lambda} \\
&= 4265.10
\end{aligned}$$

We then do the appropriate combination:

$$P^{CB} = \frac{1000}{10,000} \cdot 6755.64 + \frac{30}{500} \cdot 4265.10 = 931.47$$

One might notice that if we are just scaling each asset at the end, can't we just have assumed those payments and used the same R_I's? In other words, couldn't I have used 1000 for the calculation for the ZCB instead of 10,000 and use 30 for the calculation of the annuity instead of 500? The answer is yes! You'll get the exact same answer (and save steps)! I will say, if you aren't still fully used to this calculation, work through it step by step like we did, and then rescaling it.

Part B: Find the internal rate of return of the coupon bond. Can you solve this on an exam?

Here, we can just apply the definition:

$$P = \sum_{i=1}^{10} \frac{30}{(1 + R_I)^i} + \frac{1000}{(1 + R_I)^{10}}$$

$$\implies P = 30 \cdot \sum_{i=1}^{10} \lambda^i + 1000\lambda^{10}$$

$$\implies P = 30 \cdot \lambda \frac{1 - \lambda^{10}}{1 - \lambda} + 1000\lambda^{10}$$

$$\implies 1030\lambda^{11} - 1000\lambda^{10} - (30 + P)\lambda + P = 0$$

$$\implies 1030\lambda^{11} - 1000\lambda^{10} - 961.47\lambda + 931.47 = 0$$

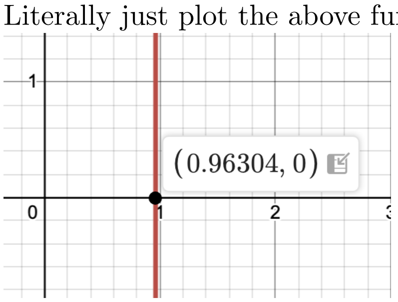
Ok the last 2 steps were a jump, but you should convince yourself that it's just algebra and rearranging terms.

This is a really large polynomial. You will not be able to solve this in an exam. However, this is a valid homework assignment, so let us go over how we solve roots:

Root Finding Methods

Method 1: Desmos (Highly Recommend)

Literally just plot the above function in Desmos, find the zeros, and you have λ:



ALWAYS show your function in your solution, as in, tell us which root finding method you used and how. A screenshot is plenty.

Common mistake: This is the value of λ. You should solve for R_I using λ.

Method 2: Bisubsection Method

This is where we get into numerical methods. You will NOT need to know any of this for this class, but it just seems cool to show. Also, I got interviews where I had to implement these methods from scratch, so who knows, this might come in handy!

Say that you know that there exists only one root and that the function is monotonically increasing, Start with an interval $[a, b]$, and let the midpoint be $m = (a + b)/2$. If $f(a) < 0$ and $f(b) > 0$, then we know that the root has to be somewhere in between. If $f(m) < 0$, when we look at the interval $[m, b]$. Otherwise, we look at the interval $[a, m]$. We then run the same search again.

The way we terminate depends on your implementation. You can either say that it stops after like 10000 iterations or when $b - a$ is really small, like 10^{-5}. The second is more commonly used.

Does this kind of make sense for our problem. Well, we are trying to find an R_I that matches our payments to the initial price spot on. If it is too big, we overshoot; otherwise, we undershoot. Hence, there only seems to be one specific R_I that works, and plotting the function tells us that it is monotonically increasing, so we should use it! The code/results are in the notebook.

Method 3: Newton's Method

This method requires the derivative of the function, and we need to start off with a good guess. We then iteratively update our guess as so:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The idea is that when we are at a point, we follow the tangent line until we hit the x-axis, which is where we then have our next guess.

This method typically converges faster than the bisubsection method. The weakness is that it requires a good initial guess, and that we need to explicitly calculate the derivative. Here, we have a pretty small range, and we are only working with a polynomial, so it should work relatively well. Again, the code is in the notebook.

If you are interested in these kinds of method, aka writing/running code on mathematics, proving why some of these methods converge, why some converge better than others, etc., take 21-369 numerical methods. Definitely one of my favorite courses at CMU.

5 Problem Session 5

Short problem session today! Unfortunately, that means a lot after we get back, but that's ok! We also go through a bit of LaTeX at the end!

Review of Internal Rate of Return

In lecture, we gave a single definition of what R_I represents. We added a few more properties that will make our lives easier in the future.

The Intuitive Definition: A constant interest rate for a specific asset

This comes directly from the definition. Basically, for a given asset, the price also comes from assuming a constant interest rate R_I and then discounting every payment.

Property 1: Different assets typically have different IRR's (unless scaled)

Consider asset A that pays $100, 200, 300, 400, 500$ at $t = 1$ to $t = 5$ and an asset B that pays $500, 400, 300, 200, 100$? No! We don't know how the interest rates will change over time, so we know nothing about how the R_I's compare.

Now let's say asset B pays $50, 100, 150, 200, 250$. Are they the same now? Intuitively, probably, since we just have the exact same payments just scaled, so something tells us in our head that the 'average' interest rate should be the same.

In fact, we can verify this to be true. By replication, $P^A = 2 \cdot P^B$. Assume that they have the same R_I. We then have that

$$P^A = \sum_{i=1}^{5} \frac{100i}{(1+R_I)^i} = 2 \cdot \sum_{i=1}^{5} \frac{50i}{(1+R_I)^i} = 2 \cdot P^B$$

which is exactly what we wanted.

Property 2: R_I is between the minimum and maximum of all spot rates

You proved this on a homework. Intuitively, this should make sense, since R_I seems to be an average of all the different interest rates.

Property 3: Consider 2 assets X and Y where we let $Z = X + Y$. we have that $\min(R_I^X, R_I^Y) \leq R_I^Z \leq \max(R_I^X, R_I^Y)$

A little harder to think about, but again you proved it on the homework, so you can take it for granted. If you think of it as having a coupon bond that is the sum of an annuity and a ZCB, since we just sum the prices, we hope you see that R_I is somewhere in between them.

Floaters and Interest Rate Swaps

First we give the definition of what $p_i[m]$ is. This is basically the interest rate from time $\frac{i}{m}$ to time $\frac{i+1}{m}$, so 1 dollar will grow to $1 + \frac{p_i[m]}{m}$. This is different from a spot rate. A spot rate is the interest rate that we set at time 0. This rate is observed at time $\frac{i}{m}$.

Consider a **floating bond** (a floater) that pays $F \cdot \frac{p_i[m]}{m}$ at each time until N, where we pay an additional F. This is now different from a regular coupon bond, since we don't know how much we will be paying each time until the actual time. How can we replicate it?

Well, let's see what happens if we start of holding F. In $\frac{1}{m}$ years, we make an interest of $F \cdot \frac{q_i[m]}{m}$, which is exactly what we wanted for the payment! We can therefore make that payment, continue holding F, and paying those values as we go. Finally, we still have F, which we will pay. This tells us that this strategy should have an initial price F, the price of a floating vessel.

Now let's consider something a bit different, at each time period, we pay $F \cdot \frac{p_i[m]}{m}$ and earn $F \cdot \frac{q[m]}{m}$ at each time until $t = N$. This is called an **interest rate swap**. How do we find a $q[m]$ to make the time-0 price of this swap 0?

The first payments should be very similar to before: Hold a floater to replicate the payments. The only problem is that we will pay an extra F. How do we make sure that doesn't happen? We are short $F \cdot d(N)$ at the beginning!

The payments we make should be trivial. It's kind of like an annuity so we just discount it, giving us

$$\sum_{i=1}^{mN} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m})$$

So now, we just equate them together, and solve for $q[m]$:

$$\begin{aligned} F - Fd(N) &= \sum_{i=1}^{mN} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m}) \\ \implies 1 - d(N) &= \sum_{i=1}^{mN} \frac{q[m]}{m} \cdot d(\frac{i}{m}) \\ \implies 1 - d(N) &= \frac{q[m]}{m} \cdot \sum_{i=1}^{mN} d(\frac{i}{m}) \\ \implies q[m] &= \frac{m(1 - d(N))}{\sum_{i=1}^{mN} d(\frac{i}{m})} \end{aligned}$$

Observe that the final answer doesn't depend on F. This kind of makes sense, since both payments back and forth depend on the scale of F.

This is where most people fall under the trap of "copying the formula on the cheat sheet and plugging in numbers on the exam". If you don't understand this replication, what we can do is create a variation, so the formula no longer works. We will go through 2 examples.

Interest Swap Variation 1

Consider the exact same thing as before, but we don't stop. Up to time $2N$, we then double the payments. In other words, from time 0 to N:

A pays B: $F \cdot \frac{p_i[m]}{m}$
B pays A: $F \cdot \frac{q[m]}{m}$

From time N to $2N$:
A pays B: $2F \cdot \frac{p_i[m]}{m}$
B pays A: $2F \cdot \frac{q[m]}{m}$

Same question as before, find $q[m]$ to make the time 0 price 0.

The strategy should be the same, replicate each person's payments, and equate them. The question is how. Consider A to B. What happens if we simply hold the floater until $2N$. That works for the first period, but for the second period, we still need $F \cdot \frac{p_i[m]}{m}$!

One idea is to just buy a floater at time N, and hold it until $2N$. In other words, have $Fd(N)$ in the bank, which becomes F at $t = N$, get a floater, and replicate the remaining payments. Similarly to before, we need to short $Fd(2N)$ for the extra face value payments. Hence, the time 0 value for the payments from A to B is given

$$F - Fd(2N) + Fd(N) - Fd(2N) = F(1 + d(N) - 2d(2N))$$

The B to A should still be trivial. It's just a longer summation:

$$\sum_{i=1}^{mN} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m}) + \sum_{i=mN+1}^{2mN} 2F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m})$$

where you then set them equal to each other and solve for $q[m]$. The derivation is left as an exercise to the reader :)

Interest Swap Variation 2

Consider the exact same thing as before, but we don't stop. Up until time $2N$, we swap who is making the payments. In other words, from time 0 to N:

A pays B: $F \cdot \frac{p_i[m]}{m}$
B pays A: $F \cdot \frac{q[m]}{m}$

From time N to $2N$:
A pays B: $F \cdot \frac{q[m]}{m}$
B pays A: $F \cdot \frac{p_i[m]}{m}$

A to B should be pretty straightforward: Hold a floater to $t = N$, and then discount the fixed payments:

$$F - Fd(N) + \sum_{i=mN+1}^{2mN} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m})$$

B to A has a similar problem, having only floaters in a later time period. We can do the exact same thing: Hold $Fd(N)$ until $t = N$, and then buy a floater. Hence, the other side has price

$$\sum_{i=1}^{mN} F \cdot \frac{q[m]}{m} \cdot d(\frac{i}{m}) + Fd(N) - Fd(2N)$$

Some people might ask the question, what if we replicated the latter by holding a floater with maturity $2N$ and shorting one with maturity N? You will get the exact same value! As a sanity check,

$$(F - Fd(2N)) - (F - Fd(N)) = Fd(N) - Fd(2N)$$

As always, we leave the algebraic derivation to the reader.

Forward Rates

We talked about forward rates before, so we will just go through a review of the basic one.

I want to price a forward contract that pays $\mathcal{F}$ at time T_d and receives F at time T_b, where $T_d < T_b$. How do I find the forward price $\mathcal{F}$? Just discount the payments and set them equal:

$$\mathcal{F}d(T_d) = Fd(T_b) \implies \mathcal{F} = F\frac{d(T_b)}{d(T_d)}$$

Let's make it a bit challenging. I will pay $\mathcal{F}$ and time t_b and receive payments F_j for times $t_j, t_{j+1}...t_N$ and $t_b < t_j$. How do I find the forward price? Same thing as before, just discount the payments and set them equal to each other:

$$\mathcal{F}d(T_b) = \sum_{i=t_j}^{t_N} F_i \cdot d(i) \implies \mathcal{F} = \frac{1}{d(T_b)} \sum_{i=t_j}^{t_N} F_i \cdot d(i)$$

Forward Rates with Multiple Payments

Let's say there's a coupon bond that pays from 1 to N, and we pay for the forward price at time i where $1 < i < N$. What should we earn?

The answer is all the payments that haven't been paid so far! Intuitively, this should make sense, if I pay later, I only deserve what if defined later. In other words, think of the entire sequence of payments as a semester course. You can join midway, and the course still went on without you, but you only pay to get what is remaining.

Simple example: Let's say there is an asset that pays F_i at each time T_i from $T_i = 1$ to $T_i = N$. Consider a forward contract such that after the jth payment is made, you pay $\mathcal{F}$ and get all the payments afterward. How do you relate the price of this asset P_0 with the forward price $\mathcal{F}$?

The idea is still replication; we want to have the asset be a combination of the forward contract and something else. What are we missing? The initial payments! Hence, discounting everything back, we have that

$$P_0 = \sum_{i=1}^{j} F_i d(T_i) + \mathcal{F} d(T_j)$$

The reason we need to discount the forward price is that that is the price you pay at time T_j to get everything after, which needs to be discounted back. During lecture, we tried a different replication strategy, but the equation will be similar to above, just in a different form.

LaTeX Tips

In the end, we started with LaTeX tips. The easiest way to get started is to go to overleaf.com, use your andrewid account, and just start coding. The more you use it, the better you become. Google is also your best friend. Almost all symbols you do not know can be found on google.

I would look at the recorded lecture for the actual workshop, so here, I will just leave the code for some common symbols that you can just copy paste:

This is how to put things in math mode: $x + y = z$
You can make it bigger and centered:

$$x + y + z$$

This is how to type fractions: $\frac{a+b}{c+d}$
You can have superscripts: R^A, subscripts: R_{IRR}, or both: R^A_{IRR}
You can bold **text** or $\mathbf{math}$
For proofs, use align*:

$$\begin{aligned} x^2 + 4x + 4 =& 0 \\ \implies (x+2)^2 = 0 \\ \implies x = -2 \end{aligned}$$

If there are diagrams, attach them as an image. Don't try to draw them in latex.

If you are new to latex, I claim that it is pretty easy to get started, especially for something like 21-270. If you have been using it already, I claim there is always something new to learn to make your solutions neater!

Have a wonderful spring break guys!

6 Problem Session 6

Hope everyone had a wonderful spring break! This time, we finished our discussion about forward contracts, then talked about put-call parity, and ended with a discussion on how to prepare for Exam 2! These 2 weeks will be kinda hefty, so be prepared!

Forward Contracts with Stocks

We have dealed with forward contracts with stocks before. If we want to buy a stock at time T for a price of $\mathcal{F}$, we can replicate that buy holding the stock and the discounted value, so

$$S_0 = d(T)\mathcal{F} \implies \mathcal{F} = \frac{S_0}{d(T)}$$

One new thing I want to say here is that you can kind of treat the forward price as the "future price" of an asset. In other words, if there exist these forward contracts, you can basically say that an asset is worth $\mathcal{F}$ at time T, and then discount it back. **This is merely for intuition sake, to calculate / prove things, you would still need to buy / sell future contracts**.

Stocks with Known Dividends

The problem with known dividends is that if A holds a stock and gets paid some money, B paying $\mathcal{F}$ would only pay for the final stock, so in a way, they don't get the same benefit as A. In the notes, the strategy would be to create a replicating portfolio with value 0 such that all the payments cancel out. I will give an alternative interpretation (use the one that makes the most sense to you), but basically we should add the dividend to the person paying the forward price, so the total value they earn is the same as the other person holding the stock.

Assuming the dividend pay δ per stock at time τ, we have that

$$S_0 = d(T)\mathcal{F} + d(\tau)\delta \implies \mathcal{F} = \frac{1}{d(T)}(S_0 - d(\tau)\delta)$$

Now, what if we have multiple payments, like δ_1 at τ_1 and δ_2 at τ_2? Well, the idea is still the same, just discount them both!

$$S_0 = d(T)\mathcal{F} + d(\tau_1)\delta_1 + d(\tau_2)\delta_2 \implies \mathcal{F} = \frac{1}{d(T)}(S_0 - d(\tau_1)\delta_1 - d(\tau_2)\delta_2)$$

We can generalize this into a sum in fact:

$$\mathcal{F} = \frac{1}{d(T)}(S_0 - \sum_i d(\tau_i)\delta_i)$$

Stocks with Unknown Dividends

Now, what if we say the stock pays αS_τ of a dividend, where we don't know when τ is, only that it's before T? Well, we cannot use the replication strategy anymore, since we can't discount something we don't know. Here, we introduce the assumption that the stock price drops as much as the dividend payment. **On an exam, you should be able to rededuce everything afterwards if we modify this assumption**.

The idea is to reinvest that dividend back into the stock immediately, so we still have some information to keep track of. Here's an example to better illustrate what we are doing: Let's say we hold one stock currently valued \$80, and $\alpha = \frac{1}{4}$. We will earn a dividend of \$20, and the stock price will drop to \$60. We reinvest that dividend into stocks, so we have \$80 worth of \$60, or $\frac{4}{3}$ shares. Our goal is to calculate how we got the extra $\frac{1}{3}$.

Let $S_{\tau-}$ be the price of the stock before the dividend pays and $S_{\tau+}$ be the price right after it drops. By definition, we have that

$$S_{\tau+} = (1-\alpha)S_{\tau-}$$

Now, let's say we start off holding Δ shares of stock. How much additional stock do we buy? The calculation is easier than people think:

$$\text{new stocks} = \frac{\text{dividend value}}{\text{new stock price}} = \frac{\Delta \cdot \alpha S_{\tau-}}{S_{\tau+}} = \frac{\Delta \cdot \alpha S_{\tau-}}{(1-\alpha)S_{\tau-}} = \frac{\Delta\alpha}{1-\alpha}$$

Hence, the total amount of stock we have is

$$\Delta + \frac{\Delta\alpha}{1-\alpha} = \frac{\Delta}{1-\alpha}$$

In other words, if we started with one stock, we would end up with $\frac{1}{1-\alpha}$ stocks at the end, so the forward price will pay more than we want! Rather, let $\Delta = 1-\alpha$. If we hold $\Delta = 1-\alpha$ stocks initially, we end up with 1 stock at the end, which is what we want! This gives us the new replicating strategy:

$$(1-\alpha)S_0 = d(T)\mathcal{F} \implies \mathcal{F} = \frac{(1-\alpha)S_0}{d(T)}$$

Now, what if we pay multiple dividends, let us say 2 for example. Well, if we initially hold $(1-\alpha)^2$ of stock, the first dividend allows us to have $(1-\alpha)$ stock, and the second allows us to have 1 stock. For N of the same payments, this would just be $(1-\alpha)^N$.

Example 1.1

Consider a dividend that pays α of the stock at σ and then pays δ per stock at time τ , where $\sigma < \tau$. What is the forward price of this stock?

Notice that we have all the tools we need! If we hold $(1-\alpha)$ stock at the beginning, then the first payment will give us a full stock, and then we have δ, using the example before! This gives us:

$$(1-\alpha)S_0 = d(T)\mathcal{F} + d(\tau)\delta \implies \mathcal{F} = \frac{1}{d(T)}((1-\alpha)S_0 - d(\tau)\delta)$$

Example 1.2

Now let's say $\tau < \sigma$, so we pay the fixed dividend first. Does the strategy/formula change?

The key idea is the δ per stock. If we start with holding $1-\alpha$ stock, then the dividend we pay is $(1-\alpha)\delta$ instead of δ. Only after the second dividend do we get a full stock. Hence, this is how the formula changes:

$$(1-\alpha)S_0 = d(T)\mathcal{F} + d(\tau)(1-\alpha)\delta \implies \mathcal{F} = \frac{1}{d(T)}((1-\alpha)S_0 - d(\tau)(1-\alpha)\delta)$$

Intuitively, in the first example, because we get paid a part of the stock first, that δ will overall pay more, so the holder of the stock will get more value. Therefore, the forward price should be smaller since they aren't benefiting as much.

Forward Contracts with Commodities

There really isn't much to talk about in this subsection. You can buy gold today, you can buy gold later, and the value of gold, if you store it, does not change over time. Typically, we don't allow shorting of commodities, and let there be storage costs for storing these commodities over a period of time. When dealing with these types of problems, make sure that you are following the strategy of buying the commodity today, holding it, paying the storage cost, and then selling it for the forward price. In this case, the intuition of "the forward price being the value of a commodity at a certain time" might make more sense.

The only inequality that really matters here is

$$S_0 + C^s_{0,T} \geq d(T)\mathcal{F}_{0,T}$$

Just to make sure that the strategy above doesn't create arbitrage. A good exercise is to prove that the other direction creates an arbitrage.

Put-Call Parity

In an earlier homework, you were asked to hold a put option and short a call option. You then calculated the payout as follows:

$$\begin{aligned} P_T - C_T &= (K - S_T)^+ - (S_T - K)^+ \\ &= K - S_T \end{aligned}$$

Under all cases. Hence, another way to replicate this portfolio is by holding $Kd(T)$ and shorting a stock. In other words,

$$P_0 - C_0 = d(T)K - S_0$$

However, this ignores the assumption of dividends. Notice that all we want is to get a stock at time T, so we introduce what we have, forward contracts! Pay $\mathcal{F}$ instead of a stock by using a forward contract! Discounting this back, this gives us the more powerful put-call parity formula:

$$P_0 - C_0 = d(T)(K - \mathcal{F})$$

Hence, using the formulas beforehand, we can relate $\mathcal{F}$ to the initial stock price and dividend information.

If you ever are given 2 out of the put option, call option, or stock price, your brain should immediately inform you that you can find the third one.

Later, you are given bounds for the initial prices for put and call options, which you need to prove in the homework. My hint to you is that in a previous homework, you have shown that if $X_T \geq Y_T$ always, then $X_0 \geq Y_0$.

Chooser Options

We skipped this during the actual session, but since it's pretty sure, I will just give a bit of a description here.

A **chooser option** basically says that at time τ, you decide whether to have a put or call option at time T. Clearly, at time τ, it's hard to look into the future, but you want to see which option has more value now, so C_τ vs P_τ. We show that they are related using put-call parity and, using some algebra, V_τ depends on call and put options valued at time τ, which we can replicate by holding them from the beginning.

The key point of this subsection is how we can use put-call parity to pretty much make a common replication strategy that we want.

Example 2.1

Say that we have a stock that pays no dividends, and we have 2 assets $V_T = \max\{S_T, K\}$ and $W_T = \min\{S_T, K\}$. Prove $V_0 = S_0 + P_0$.

This idea comes directly from replication. We have that

$$\begin{aligned} V_T &= S_T + \max\{0, K - S_T\} \\ &= S_T + P_t \\ \implies V_0 &= S_0 + P_0 \end{aligned}$$

Example 2.2

Now prove that $W_0 = S_0 - C_0$

This is a bit trickier. W has a min, but C is a max. What do we then relate? Well, there's this really neat trick:

$$max(a, b) + min(a, b) = a + b$$

which gives us a way to directly substitute a maximum with a minimum!

$$\begin{aligned} W_T =& \min\{S_T, K\} \\ &= S_T + K - \max\{S_T, K\} \\ &= S_T + K - (K + \max\{S_T - K, 0\}) \\ &= S_T + K - (K + C_T) \\ &= S_T - C_T \\ \implies W_0 &= S_0 - C_0 \end{aligned}$$

Example 2.3

Finally, prove that $V_0 + W_0 = S_0 + Kd(T)$

I claim that there are 2 ways to approach this. The first way involves adding our results above and using put-call parity:

$$\begin{aligned} V_0 + W_0 &= 2S_0 + P_0 - C_0 \\ &= 2S_0 + (Kd(T) - S_0) \\ &= S_0 + Kd(T) \end{aligned}$$

The second way is even simpler. $V_T + W_T$ is contains both the maximum and the minimum, so it equals $S_T + K$. Discounting it back gives us exactly what we need.

7 Problem Session 7

Hey guys, really short problem session today! Today we discuss my favorite and least favorite topics, **American Options** and **Risk Neutral Pricing**! You get to decide which is which ;)

American Options

Recall an **American option** is exactly like a European option, except the holder can exercise it earlier at some time $t \le T$. I would like to emphasize the importance of "the holder" in this definition. If you are long an American option, YOU get to decide when to execute it. If you are short one, someone else has the option, so THEY get to decide when to execute it. This will play an important role for our proofs later.

For this class, the only thing you should care about is that things with more options typically cost more. If 2 assets only differ by having an extra option, that extra option could be a way out, so it should cost more. **We never have to think about how the market moves**! People tend to over complicate American options, thinking that we might exercise at a bad time etc., we don't care! We cannot predict how the market changes in the future, we can only deduce prices based off of replication!

There really only are 3 important properties of American options:

1. American options with higher maturity are more expensive: $V_t^{A,T_1} \le V_t^{A,T_2}$ if $T_1 \le T_2$. Intuitively, this makes sense since more options means more expensive asset. Proof-wise if we long the option with a longer maturity and short the one with a shorter maturity, whenever the shorter one gets executed, we can always match it with our longer option.

2. The price of an American option is always greater than its intrinsic value: $V_t \ge g(S_t)$, where $g(S_t)$ basically means how much money you make if you exercised it right now. Same intuitive definition, having more options in the future causes the value to be greater than just executing it today.

3. The price of an American option is greater than a European option: $V_t^A \ge V_t^E$. Same intuitive reasoning, more choices = more money. Proof-wise, consider longing the American option and shorting the European option. We are currently longing for the American option, so we choose the strategy to wait until maturity. This always guarantees us that the

payments will cancel out. If $V_t^A - V_t^E < 0$, then we would have extra money in the bank to create an arbitrage. Therefore, the above inequality has to follow.

One more conclusion I would like to make: In class, we showed that if $R > 0$, then

$$C_t^A \geq C_t^E > S_t - K$$

where the first inequality we just explained above comes from algebra (check the notes). This implies that

$$C_t^A > S_t - K$$

Where the right side term represents the value of executing the option at that time. Since the value of the option itself is worth more, we would never choose to execute it at that time. Hence, we would always hold it.

Since we would never execute it early, we would always wait until maturity. This means that the American option is basically a European option, so $C_t^A = C_T^E$. Make sure you understand this derivation, you will do something similar in the homework.

Proofs with America Options

Most American option proofs with inequalities will typically follow the same 3 steps:

1. Move all the terms to one side to have some form $X_t \geq 0$. You will then AFSOC (assume for the sake of contradiction) that $X_t < 0$.

2. Show that X_T can always make money. If you are longing an American option, pick a strategy to make money. Most of the time it will be waiting to expiration. If you are shorting one, then show that you can always match that payment whenever it is executed.

3. Since you never lose money, we have an arbitrage since $X_t < 0$. Therefore, $X_t \geq 0$.

We've seen this example with property 3 when we long an American option. We will now see an example using the other case:

Example 1.1

Consider an American call C_t, put option P_t, and straddle option V_t. Recall that a straddle option pays $|S_t - K|$. Prove $V_t \leq C_t + P_t$.

Step 1, we have

$$C_t + P_t - V_t \geq 0$$

We then want to show that this always makes a profit. Since we short an American option, we have to prove that regardless of when V_t is executed, we can always match it.

Let's say V_t is executed when $S_t > K$, so we lose $S_t - K$. Well, let's also execute our call option to make that money back! We will then have a put option that guarantees us that we will never lose money!

What about the other way? We lose $K - S_t$ when $S_t < K$, so we execute the put option and keep our call option.

Therefore, it must be the case that $C_t + P_t - V_t \geq 0$, which is what we want to prove.

Example 1.2

Show that

$$V_t \geq |S_t - \frac{K}{(1+R)^{T-t}}|$$

This looks horrible, but we can break it down. Recall using our algebra rules, we can split the inequality into 2:

$$V_t \geq S_t - \frac{K}{(1+R)^{T-t}}, \ V_t \geq \frac{K}{(1+R)^{T-t}} - S_t$$

We can prove each of them separately. Let's work with the first one. Moving everything to the left, we have

$$V_t - S_t + \frac{K}{(1+R)^{T-t}} \geq 0$$

We are currently longing the option, so I say wait until maturity and then execute it. Hence, at maturity, the value becomes

$$V_T - S_T + K$$

If $S_t < K$, the value above becomes 0. If $S_t \geq K$, we have $2(S_t - K)$. Hence, we always have a chance to make a profit, so the inequality follows! We leave the other inequality as an exercise to the reader.

Risk Neutral Probabilities

Nice **video** on intuition for risk-neutral probabilities!

Consider a simple one period model with only one stock and one bank. I claim that every strategy can be defined by the tuple (X_0, Δ_0), representing how much the initial portfolio is worth and how much stock to hold at the beginning. The rest of the money goes into the bank. In 21-370, when dealing with multiple periods, Δ just becomes a vector, in other words, by a different amount of stocks each time.

Now let's say the stock initially costs S_0, and there is a universal coin flip such that if it flip heads, the price jumps to uS_0, and dS_0 if tails. I claim the probability of heads or tails will not matter at all in our replication, we still don't need to deal with real life probabilities in this class.

In class, we showed an important property, $d < 1 + r < u$ iff there is no arbitrage. The forward direction cases in all possible tuples, whether Δ_0 is positive or negative, to show that there is always no arbitrage strategy. The backwards direction uses proof by contradiction using different examples.

Consider portfolio X_0. I will re-write this as

$$X_0 = (X_0 - \Delta_0 S_0) + \Delta_0 S_0$$

which seems useless at first, but recall that the first term represents the value in the bank, and the second term is in stocks. At time 1, this becomes

$$X_1 = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 S_1$$

which we can rewrite as

$$X_1(H) = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 u S_0$$
$$X_1(T) = (X_0 - \Delta_0 S_0)(1 + r) + \Delta_0 d S_0$$

Now, let's saw I have a specific payout I want to satisfy, aka if I lands heads I get some value, if I land tails I get a different value, so I'm given $X_1(H)$ and $X_1(T)$. S_0, r, u, d are all known, so the only unknowns are X_0 and Δ_0. We have 2 equations, so we can solve them! In lecture, we use algebra to figure out that

$$X_0 = \frac{1}{1+r}[(\frac{1+r-d}{u-d})X_1(H) + (\frac{u-1-r}{u-d})X_1(T)]$$

If we let $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-1-r}{u-d}$, this simplifies to

$$X_0 = \frac{1}{1+r}[\tilde{p}X_1(H) + \tilde{q}X_1(T)]$$

$\tilde{p}$ and $\tilde{q}$ are our **risk-neutral probabilities**. They have **NOTHING** to do with real-life probabilities, or how likely things are. They are just numbers between 0 and 1 corresponding to different events, and they add up to 1. They have the properties of a probability, but they are not likelihoods. Do not get them confused.

Since we are multiplying probabilities by values, we typically write this also as

$$X_0 = \frac{1}{1+r}\tilde{E}(X_1)$$

What does this mean? This means that if you tell me how the stock changes over time, I can give you the price for ANY payment outcomes. Remember how earlier I gave you a stock and did a bunch of replication things to show that almost everything can be replicated? This is exactly that!

Remember Δ_0? We also have a formula for that:

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}$$

In other words, given the payouts, I can not only tell you how much the portfolio should be worth, but also what the replication strategy is using stocks!

This is a very powerful tool, which we will see more about in this part of the course. Good luck with the homework and have fun at carnival!

8 Problem Session 8

Rip our first asynchronous problem session since the homework is due really soon. Hopefully this handout benefits some people!

Pricing Measures

Recall that a **probability measure** is simply a mapping of events to numbers. We can say $\mathbb{P}(H) = 0.5$ and $\mathbb{P}(T) = 0.5$, and then $\mathbb{Q}(H) = 0.2$ and $\mathbb{Q}(T) = 0.8$ for the same coin! We are just mapping the event of mapping heads and tails to different numbers. The only condition for probability measures is that the numbers are between 0 and 1 and they add up to 1!

With that, a **pricing measure** $\tilde{\mathbb{P}}$ is just a probability measure with 2 more properties:

1. $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$
2. $S_0^i = \frac{1}{1+r}\mathbb{E}^{\tilde{\mathbb{P}}}(S_1^i)$ for all i

It's just a definition. The second one you saw something similar last week, where we let $\tilde{p}$ and $\tilde{q}$ be the risk-neutral probabilities. If you want to prove something is a pricing measure, you just need to show that the 2 properties above hold.

Fundamental Theorem of Asset Pricing

There's 2 of them! Before stating them, let's bring in some intuition.

Remember that exercise we always do where I gave you a stock with 2 different outcomes, a bank, and showed that if you give me the payout of any 2 outcomes, I can price it using the stock and the bank? It seems like there is always only one solution. We saw the same case where there are 3 different outcomes, 2 different assets, and we can always price the 3rd asset.

Now what if we had 3 outcomes and only one asset. Well, mathematical intuition tells us that we might not be able to replicate in the same way. Something along the lines of 3 equations and 2 variables creating conflict?

What about 2 outcomes and 2 different assets? Same intuition, we only need 1 asset to create every price, so maybe there are infinitely many solutions?

Our suspicions are confirmed by the two fundamental theorems of asset pricing.

Theorem 1: The model is arbitrage-free iff there is at least one pricing measure $\tilde{\mathbb{P}}$.

Theorem 2: The model is complete iff there is exactly one pricing measure $\tilde{\mathbb{P}}$.

Let's take a second to digest each theorem. The first theorem says that if you can find one solution to $\tilde{p}$ and $\tilde{q}$ etc., then there is no arbitrage and vice versa. That kind of makes sense. Recall that we got those risk neutral probabilities through replication and matching outcomes, so if we can do that for all the assets, it makes sense for the model to be arbitrage free.

For the second, recall that a **complete** market means that every asset is replicable. This is a stronger argument than the first theorem. Therefore, we need a stronger condition, having only one pricing measure. This will make more sense when we prove the theorem, but for now, just think that a unique pricing measure can basically make you do anything.

Notice how strong these theorems are. We have a really nice formula that relates risk-neutral probabilities relating outcomes to initial prices. Hence, what we can do is assume there is a risk neutral probability measure, and depending on whether there is a solution, or infinitely many solutions, we can determine whether there is arbitrage in the market or if it is complete! We can also find values to force a specific pricing measure to make it arbitrage-free/complete! I think it's best to view this in an example:

Example 1

Consider a one-period double binomial model with a bank $r = 0.25$, two stocks, and some other securities described below. The value of the stocks at time $t = 1$ will be

$$S_1^1(HH) = 180,\ S_1^1(HT) = 180,\ S_1^1(TH) = 60,\ S_1^1(TT) = 60$$

$$S_1^2(HH) = 120,\ S_1^2(HT) = 60,\ S_1^2(TH) = 120,\ S_1^2(TT) = 60$$

You are not able to directly trade the stocks at $t = 0$ and the values S_0^1 and S_0^2 are unknown to you. The securities you can trade are the following.

1. The Arrow-Debru security V^{HH}, which pays \$1 in state HH and \$0 in all other states. The initial price of this security is $V_0^{HH} = \frac{2}{15}$.
2. The Arrow-Debru security V^{TT}, which pays \$1 in state TT and \$0 in all other states. The initial price of this security is $V_0^{TT} = \frac{4}{15}$.
3. A security W, which pays $W_1(\omega) = \max\{S_1^1(\omega), S_1^2(\omega)\}$ at time $t = 1$. The initial price of this security is $W_0 = 100$.

Is the market arbitrage free? Is it complete?

Didn't really cover Arrow-Debru securities, but they are really simple. They pay 1 dollar if they land on an event and 0 otherwise. They are extremely useful in later proofs.

To answer these questions, we need to see whether there is a pricing measure, or infinitely many. Recall by definition that a pricing measure has the property that the expected value of the outcomes has to equal the initial price discounted back. Hence, assume there is a pricing measure, let there be risk neutral probabilities $\tilde{p_1}$, $\tilde{p_2}$, $\tilde{p_3}$, $\tilde{p_4}$ for each event, and let's try to solve them:

$$\tilde{p_1} + \tilde{p_2} + \tilde{p_3} + \tilde{p_4} = 1$$

$$1\tilde{p_1} + 0\tilde{p_2} + 0\tilde{p_3} + 0\tilde{p_4} = (1+r)\frac{2}{15}$$

$$0\tilde{p_1} + 0\tilde{p_2} + 0\tilde{p_3} + 1\tilde{p_4} = (1+r)\frac{4}{15}$$

$$180\tilde{p_1} + 180\tilde{p_2} + 120\tilde{p_3} + 60\tilde{p_4} = (1+r)100$$

Now we just need to solve for this! Algebra gives us that

$$\tilde{p_1} = \frac{1}{6},\ \tilde{p_2} = \frac{1}{4},\ \tilde{p_3} = \frac{1}{4},\ \tilde{p_4} = \frac{1}{3}$$

is the only solution. Therefore, using our 2 theorems, the model is arbitrage free and complete! Hypothetically, what would happen if there were a conflict where there are more equations than unknowns? We then claim there is arbitrage in the market (yes, you will be able to find an arbitrage strategy). What if there are infinitely many solutions due to free variables? Then the market is incomplete!

To finish the problem, let's price S_0^1 and S_0^2! We will do the first one, and leave the second as an exercise to the reader:

$$S_0^1 = \frac{1}{1+r}\tilde{\mathbb{E}}[S_1^1]$$

$$
\begin{aligned}
&= \frac{4}{5}[180\frac{1}{6} + 180\frac{1}{4} + 60\frac{1}{4} + 60\frac{1}{3}] \\
&= 88
\end{aligned}
$$

The main takeaway from this mini-problem session? You can directly solve for risk neutral probabilities measures by setting a system of equations, and the solution tells you everything about the market!

9 Problem Session 9

Since people asked for it, here is the handout (even though there wasn't much to talk about).

For this exam, if you really think about it, there are only 2 topics: American Options and Risk-Neutral Probabilities. There really aren't that many formulas to remember either. As long as you fully understand the concepts and apply them, instead of just putting numbers, you will do fine.

American Options

Recall our 2 main properties:

1. American options with higher maturity has higher prices
2. American options cost more than its intrinsic value
3. American options cost more than European options

All of them fall under the same idea of "if you have more choices, things will be more expensive."

As for the proofs, it will almost surely follow the same strategy. You will move everything to one side, show that the payout is always positive, so the initial price of the portfolio will be positive too. In order to show that the payout is always positive, it kind of depends on whether you are longing it or shorting it.

If you are longing an American option, YOU ARE IN CONTROL. Pick a strategy to make money. The easiest / most common way will be to wait until maturity. If you are short, YOU ARE NO LONGER IN CONTROL. You need to consider the cases where an enemy can exercise it early or at maturity, and you should have a strategy to counteract both.

For a specific example, look back at problem session 7.

Risk-Neutral Probabilities

They have 3 properties:

1. They are greater than 0
2. They sum to 1
3. $V_0 = \frac{1}{1+r}\tilde{E}[V_1]$

In order to show something is a valid risk-neutral probability measure, that's all you need to show.

You also know 2 theorems. No arbitrage iff there exists a RNPM, complete if there exists a unique RNPM. This implies that for these problems, you can assume that there exists such a measure and then try to solve for it. If there is a solution, then you just showed that there is no arbitrage. If there is a unique solution, then you have shown that the market is complete.

Again, for a more specific example, look at the handout for problem session 8. Other than that, problem 2 on Exam 3 review problems is good to follow, as well as the **proof** for the 2nd fundamental theorem. If you can fully understand those, you are good to go.

Good luck on the exam y'all!

10 Problem Session 10

Last problem session! Today we went over the last few concepts for the course, and then did a quick review!

Remember how I said that you should never care about real life probabilities until the end of the semester? This is because everything we had dealt with was only replication and matching outcomes. Now they matter because we care about how much risk people are willing to take. This was never a problem beforehand.

Utility Functions

A **utility function** basically takes in different pay-offs and produces new numbers. The idea is that they map to how much "happiness" people have. In other words, we want a way to say that risking a \$1 coin toss is worth differently from a $\$10,000$ coin toss despite having the same expected value.

$U(x)$ is always increasing. The more money you get paid, the happier you are. U'' is typically negative. The difference between 10 and 20 dollars should be worth more than 1,000,000 and 1,000,010 dollars.

Our problem is therefore the following: Given a utility function, what is the optimal portfolio to maximize our utility? In other words find the optimal $\hat{X}_1$ such that,

$$E[U(\hat{X}_1)] \geq E[U(X_1)]$$

for all X.

The proof is complicated. For this course, all you need to know is the formulas to solve this problem. You are given a theorem that $\hat{X}_1$ maximizes utility iff there exists a λ such that

$$U'(\hat{X}_1(\omega)) = \lambda \frac{\tilde{P}(\omega)}{P(\omega)}$$

for all ω. This means that what we can do is solve that equation, and the $X_1\hat{(}\omega)$ will define our optimal portfolio. The question is how do we solve for λ? Well, every portfolio still has to follow this formula:

$$\hat{X}_0 = \frac{1}{1+r}\tilde{E}[\hat{X}_1]$$

Example 1

Consider a one-period binomial model with $u = 2$,$d = \frac{1}{2}$, $r = \frac{1}{4}$, $S_0 = 100$, $P(H) = \frac{3}{4}$, $P(T) = \frac{1}{4}$. An investor has a utility function $U(x) = \ln(x)$ and initial capital 100. Find the optimal strategy.

Looking at our theorem, we should probably solve for $\tilde{P}(\omega)$ first, which we calculate to be $\tilde{P}(H) = \tilde{P}(T) = \frac{1}{2}$. Knowing the fact that $U'(x) = \frac{1}{x}$, we have that

$$\frac{1}{X_1(H)} = \lambda \frac{\tilde{P}(H)}{P(H)} \implies \frac{1}{X_1(H)} = \lambda \frac{1/2}{3/4} \implies X_1(H) = \frac{3}{2}\frac{1}{\lambda}$$

$$\frac{1}{X_1(T)} = \lambda \frac{\tilde{P}(T)}{P(T)} \implies \frac{1}{X_1(T)} = \lambda \frac{1/2}{1/4} \implies X_1(T) = \frac{1}{2}\frac{1}{\lambda}$$

The risk neutral pricing formula gives us that

$$\begin{aligned}
(1+r)X_0 &= \tilde{E}[X_1] \\
&= \frac{1}{2} \cdot (\frac{3}{2}\frac{1}{\lambda}) + \frac{1}{2} \cdot (\frac{1}{2}\frac{1}{\lambda}) \\
\implies 1.25 \cdot 100 &= \frac{1}{\lambda} \\
\implies \frac{1}{\lambda} &= 125
\end{aligned}$$

We'll keep it as $\frac{1}{\lambda}$ since we will directly substitute it into $X_1(\omega)$. Putting them together, we have $X_1(H) = \frac{375}{2}$ and $X_1(T) = \frac{125}{2}$. Feel free to use whatever system of equations/pricing method to figure out the exact strategy from here!

Example 2

We will go over another example where we will practice identifying problems as certain types without explicitly revealing the parameters.

You are at a casino, and you can bet on any number from 1 to 6 from a dice roll. The price of a bet is 1/6, and you get 1 if it lands on the same number as your bet. You can also place fractional bets.

It turns out that through observation you feel like the casino is rigged and the dice are not fair. Rather, here are the probabilities that you see:

$$1 : 20\%,\ 2 : 18\%, 3 : 16\%, 4 : 13\%, 5 : 18\%, 6 : 15\%,$$

You have a utility function of $U(x) = -e^{-x}$ and 1 dollar initially. Assume $r = 0$. How do you play to maximize your utility?

You know this is a utility problem since you are given a utility function. You know $U(x)$ and $P(\omega)$ from the problem. How do you solve for $\tilde{P}(\omega)$. Well, consider betting on 1. The price is 1/6 and you get 1 if you land on ω_1. This will give you $\tilde{p_1} = \frac{1}{6}$. This generalizes to all of the $\tilde{p}$. You will then follow a similar method as above, setting up 6 equations, solving for λ, etc. etc.

Mean Variance

The idea is that we might want to achieve a specific rate of return even if there is risk involved. Hence, given a specific return, we want to find a portfolio that minimizes the variance. We first denote some notation:

$$\mu_i = \frac{E[S_1^i]}{S_0^i} - 1,\ \sigma_{ii} = \frac{Var[S_1^i]}{(S_0^i)^2},\ \sigma_{ij} = \frac{Cov[S_1^i, S_1^j]}{S_0^i S_0^j}$$

Although these formulas are not directly given in the notes, where the notes use ρ, you should know how to derive them.

Again, we are given the theorem that if we have a $\hat{x}^j$ value of stock j at time 0, it minimizes the variance iff there exists λ such that

$$\sum_{j=1}^{k} \sigma_{ij}\hat{x}^j = \lambda(\mu_i - r)$$

for all i. In this case, we don't want to use the risk neutral probability formula, using the fact that we want each individual value to grow on average as much as our wanted $\hat{r}$, we have that

$$\sum_{j=1}^{k} (\mu_j - r)x^j = (\hat{r} - r)X_0$$

The calculations will be quite similar to the utility function, solving for values in terms of λ using the system of equations, solving for λ using the final equation and substituting it back in. The main difference is that, rather than computing the outcome values, we will be computing the amount of each value to hold initially.

If the assets are all uncorrelated, that is, all of them have 0 covariance, notice that the summation simplifies to σ_{ii} as all the σ_{ij} are 0. This simplifies to give us

$$\hat{x}_i = \frac{\lambda(\mu_i - r)}{\sigma_{ii}}$$

and thus

$$\lambda = \frac{(\hat{r} - r)X_0}{\sum_{j=1}^{k} \frac{(\mu_j - r)^2}{\sigma_{jj}}}$$

This shortcut only works if they are not correlated. **DO NOT USE IT IF THE COVARIANCE IS NON-ZERO!!!**

Observe intuitively why this formula works. When $\mu_i > r$, aka we have a greater return than r, we should long the asset. If the variance is really big, we should not hold as much.

Example 2

Let $r = 0.1$, $S_0^1 = S_0^2 = 100$, $E[S_1^1] = 115$, $E[S_1^2] = 120$, $Var[S_1^1] = 1000$, $Var[S_1^2] = 2000$, $Cov[S_1^1, S_1^2] = -1000$. Also, $X_0 = 1000, \hat{r} = 0.12$. Solve for $\hat{x_1}, \hat{x_2}$.

We first start off by converting them into the values we need:

$$\mu_1 = \frac{E[S_1^1]}{S_0^1} - 1 = 0.15$$

$$\mu_2 = \frac{E[S_1^2]}{S_0^2} - 1 = 0.2$$

$$\sigma_{11} = \frac{Var[S_1^1]}{(S_0^1)^2} = 0.1$$

$$\sigma_{22} = \frac{Var[S_1^2]}{(S_0^2)^2} = 0.2$$

$$\sigma_{12} = \frac{Cov[S_1^1, S_1^2]}{S_0^1 S_0^2} = -0.1$$

Again, we start off setting up the system of equations:

$$\sigma_{11}\hat{x_1} + \sigma_{12}\hat{x_2} = \lambda(\mu_1 - r) \implies 0.1\hat{x_1} - 0.1\hat{x_2} = 0.05\lambda$$

$$\sigma_{21}\hat{x_1} + \sigma_{22}\hat{x_2} = \lambda(\mu_2 - r) \implies -0.1\hat{x_1} + 0.2\hat{x_2} = 0.1\lambda$$

Solving this gives us that $\hat{x_1} = 2\lambda$, $\hat{x_2} = 1.5\lambda$. Plugging this into the later equation, we have that

$$(\mu_1 - r)\hat{x_1} + (\mu_2 - r)\hat{x_2} = (\hat{r} - r)X_0 \implies 0.1\lambda + 0.15\lambda = 20 \implies \lambda = 80$$

Which gives us that $\hat{x_1} = 160$, $\hat{x_2} = 120$. The calculation directly tells us to buy 1.6 of S^1, 1.2 of S^2, and put the rest in the bank.

11 Final Review Checklist

Here is a comprehensive checklist to see if you understand 21-270 content:

1. Given a portfolio of stocks, the bank, and options, can you draw the payout diagram?
2. What is arbitrage? What is the full definition of arbitrage? What does it mean for the market to have no arbitrage? How do we use it in proofs?
3. Why do we place so much emphasis on replication? How does replication work? Can you set up a system of equations to solve it?
4. What are nominal and effective interest rates?
5. What is the spot rate? When do you use it?
6. What are forward rates? How do you find them?
7. Why do we use discount factors
8. What is a zero coupon bond, annuity, and coupon bond? How do we calculate the time-0 price? How can we replicate the other if we know 2 of them?
9. What is the internal rate of return? How do you calculate it for the above assets? What are some inequalities you know about them?
10. What's a floating bond? Why is the price F?
11. What's an interest rate swap? Can you derive the formula for different variations?
12. What's a forward contract? How do we calculate the forward price for fixed payments?
13. How do we calculate the forward price for stocks with fixed dividends, with dividends that are a fraction of the stock price?
14. How do forward contracts work with commodities? What's the storage cost?
15. What is put-call parity? Why do we use it?
16. What is a chooser option?
17. What are the main properties of American options? How do you use them in proofs?
18. What are risk-neutral probability measures? What properties do they have?
19. What are the first and second fundamental theorems of asset pricing? What do they allow us to do?

20. What are Arrow-Debu securities? Why do we use them?
21. What are utility functions? How do you find the portfolio that maximizes utility?
22. How do we find the optimal portfolio that minimized variance for a specific return?

12 So what now?

Congratulations! You made it to the end of 21-270! Some of you might be interested in what the following math finance courses would be about, so here is just a taste of what you might expect. I was a TA for 21-420. I was NOT a TA for 21-370 or 21-378, and the content can vary by semester.

21-370

- In 21-270, you dealt with the one-period model, where you can calculate risk-neutral probabilities to find the time-0 price. What do you do when there are more time periods? Use the same formula at time n to find the time $n-1$ price, and so on!
- You will dive deeper in probability theory, mainly conditional probability, martingales, random walks, etc.
- We talked about American options, but skipped over how we make decisions on when to execute. With a multi-period model, we can analyze this.
- What if we didn't have a fixed length, but just say stop when our portfolio reaches a specific value?
- We mentioned how we can find an optimal strategy to maximize our utility. Is this the same when there are multiple periods?
- You will see the Black-Scholes model!

21-378

- Remember chapter 2 where we dealt with interest rates and the ZCB/annuity/CB combo? We will go into this really in depth.
- You will dive deeper into how IRR's relate to different interest rates
- You will learn about DV01/Duration, which measures how sensitive prices change with regards to the IRR.
- You will see how we can see how much the price depends on different interest rates through sensitivity analysis. Hedging will come into play since difference prices depend on the same interest rates, which will get into principal components.
- You will see how we can model interest rates as a function of time.

- You will also see backward induction here, but will also use spreadsheets to calculate the initial price. Hence, you can calculate the initial price of American options as well as optimal execution strategies.

21-420

- You will start of reviewing probability stuff like law of large numbers/central limit theorem. You will then immediately jump into stochastic calculus.
- You will dive deep into Brownian motion, understand their relation to martingales, and then learn what Ito integrals are (basically instead of dt you have dW_t).
- You will then learn Ito's formula, how we model the stock price with Brownian motion, to give us the Black-Sholes formula.
- You will then see how we can derive something similar if some asset depends on many different variables, maybe multiple stock prices, or a stock price depending on 2 different random walks.
- You will see how risk neutral probabilities can be constructed for this continuous-time case
- You will end by seeing how we apply all of this to the fundamental theorems of asset pricing, dividend paying stocks, forwards/futures etc.

Final Remarks

Thank you to Professor Handron for giving me the opportunity to lead these problem sessions and the flexibility to shape these however I see fit, Professor Hrusa for his wonderful notes that are still used for the 21-270 course to this day, to all of y'all who come to my problem sessions/office hours, and to everyone who has watched one of my videos or seen any of my handouts. I had a great time creating all this material and presenting it, and I hope I made even a minimal impact on your learning this semester.

Good luck on the exam, get some sleep, and make good life decisions!